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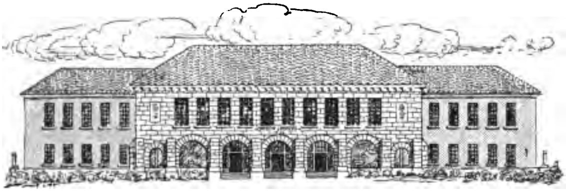

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NEW

ELEMENTARY ALGEBRA

DESIGNED FOR

COMMON AND HIGH SCHOOLS
AND ACADEMIES.

BY

SHELTON P. SANFORD, A.M.,

Professor of Mathematics in Mercer University, Ga., and Author of
a Series of Analytical Arithmetics.

PHILADELPHIA:
J. B. LIPPINCOTT COMPANY.

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PREFACE.

In the work now offered to the public the author hopes that he has succeeded in making algebraic science so clear and simple that the ordinary student will find it not only easy to understand, but attractive.

Special attention has been bestowed on the subjects of Notation, Simple Equations and Radical Quantities, and it is believed that the treatment is an improvement on former methods.

The book is merely introductory, but it embraces a range of subjects sufficiently extensive to meet the wants of pupils in Common and High Schools and Academies, and to furnish ample preparation for the full and exhaustive treatises used in the higher institutions of learning. S. P. S.

MERCER UNIVERSITY,
MACON, GA.

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ELEMENTS OF ALGEBRA.

SECTION I.

DEFINITIONS AND NOTATION.

1. **MATHEMATICS** is the science of quantity.

2. **QUANTITY** is anything that can be measured; time, distance, weight, money, number, etc. are quantities.

3. **ARITHMETIC** is a branch of Mathematics in which quantities are represented by certain characters called *figures*, as 1, 2, 3, etc.

4. **ALGEBRA** is a branch of Mathematics in which quantities and their relations and properties are made the subject of calculation by means of *letters* and other *symbols*.

5. Algebra had its origin in necessity. As there were many classes of questions that Arithmetic could not satisfactorily solve, Algebra was invented for the purpose of solving such problems.

The object of Arithmetic and of Algebra is the same— viz. to investigate the relations and properties of quantities. In many respects the processes of both are the same, whilst in other respects they are quite different. Algebra carries its researches much farther than Arithmetic; hence it was called by Newton *Universal Arithmetic*.

NOTATION.

6. **NOTATION** is the art of *writing* quantities that are the subject of calculation. In Arithmetic we represent quantities by means of certain characters called *figures*, as 1, 5, 20, etc. This is called *Arithmetical* Notation.

In Algebra we represent quantities by the letters of the alphabet, as, a, b, c, . . . x, y, z, etc. This we call *Algebraic* Notation.

The principal symbols employed in Algebra may be divided into two classes:

I. Those which express quantities.

II. Those which denote the relations which quantities bear to each other, or the operations which are to be performed on them.

SYMBOLS OF QUANTITY.

7. Quantities in Algebra are usually represented by letters; sometimes they are expressed by Arabic numerals, just as in Arithmetic.

NOTE.—Quantities might have been represented by any arbitrary symbols whatever, as □, △, ○, etc., but mathematicians, by common consent, have adopted *letters*, as being more easily written, and less liable to confusion in the writing.

8. For convenience, Arabic numerals and the first letters of the alphabet, as a, b, c, etc., usually stand for *known* or *given* quantities; while the last letters, x, y, z, etc., represent *unknown* or *required* quantities.

SIGNS.

9. In addition to the symbols of *quantity* mentioned above, Algebra employs certain symbols to express *operations* and *relations*, called SIGNS.

10. An erect cross, +, called *plus*,* indicates *addition*. Thus, $6+4$ means that 6 and 4 are to be *added* together, and is read 6 *plus* 4, or 6 *increased* by 4; $a+b$ denotes that the quantity indicated by b is to be *added* to the quantity indicated by a.

---

* *Plus* is a Latin word, meaning *more*.

11. Subtraction is indicated by a short horizontal line $-$, called *minus*,[*] written between two quantities. Thus $12-8$ signifies 12 *minus* 8, or 12 *diminished* by 8. In like manner $a-b$ signifies a minus b, or a diminished by b.

12. The sign $\backsim$ is sometimes employed to denote the difference of two quantities when it is uncertain which is the greater of the two. Thus $a \backsim b$ signifies the difference between a and b, without determining whether b is to be subtracted from a, or a from b.

NOTE.—The quantities connected by the signs $+$ and $-$ are called TERMS.

13. Multiplication is indicated by an inclined cross, $\times$, somewhat like a Roman X. Thus 12×15 indicates that 12 is to be multiplied by 15, and is read 12 *into* 15, or 12 multiplied by 15.

In like manner $a\times b$ signifies a *into* b, or a *multiplied* by b.

Multiplication is sometimes indicated by placing a point (.) between the several quantities; thus, $a.b.c.d$ expresses the same thing as $a\times b\times c\times d$. Most frequently the multiplication of quantities is indicated by writing the letters in succession without any sign between them. Thus, $abcd$ is the same as $a\times b\times c\times d$. Numerals, of course, require the sign of multiplication between them.

14. Quantities multiplied together are called *factors*.[†]

15. The sign of division is a short horizontal line with a dot above and another below, $\div$; thus, $15\div5$ signifies that 15 is to be divided by 5, and is read 15 *divided* by 5.

Most usually, division is denoted by a line, with the dividend above and the divisor below it; thus, $\frac{10}{3}$ denotes that 10 is to be divided by 3; $\frac{a}{b}$ signifies that a is to be divided by b.

---

[*] *Minus* is a Latin word, meaning *less*.

[†] *Factor*, from the Latin *facio, to make, to produce*, because, being multiplied together, they *make* the product.

16. EQUALITY is indicated by two short parallel lines, =. Thus, $6+4=10$ signifies that the sum of 6 and 4 is equal to 10; $a-b=x$ denotes that the difference of a and b is equal to x.

17. INEQUALITY is indicated by two lines meeting at a point, >. The opening is always toward the greater of the two quantities. Thus, $a>b$ denotes that a is *greater* than b; $m<n$ signifies that m is *less* than n.

18. A PARENTHESIS ( ) or a VINCULUM[*] $\overline{\quad}$ is used to denote that certain quantities connected by the signs + or − are to be subjected to the same operation. Thus $(8+4)5$ means that the sum of 8 and 4 is to be multiplied by 5, making 60. Without the parenthesis, $8+4\times5$ would indicate that 8 is to be added to the product of 4 multiplied by 5, making the result 28. $(a+b)(c-d)$ denotes that the sum of a and b is to be multiplied by the difference of c and d.

19. The student must not forget that these signs are abbreviations for words. They constitute what is called *algebraic language*, and are a species of *short-hand* by which complicated questions that in ordinary language would occupy a large space can be very briefly expressed. But in order to avail himself of the advantages of this abbreviated language, the student must first learn to translate common language into algebraic, and, conversely, to change algebraic language into common.

---

[*] *Vinculum* (Latin), meaning *chain*. Its effect is, as it were, to *bind* quantities together.

We will give a few examples to render the student familiar with the use of the symbols thus far given.

Let the student translate the following algebraic expressions into common language:

Ex. 1. $\frac{a+b}{3}+\frac{m-n}{4}=c-d.$

Ans. The sum of a and b, divided by 3, increased by the difference of m and n, divided by 4, is equal to c diminished by d.

Or, we may vary the translation thus:

One-third of the sum of a and b, increased by one-fourth of the difference of m and n, is equal to c diminished by d.

NOTE.—The student cannot fail to notice the superior conciseness and clearness of the *algebraic* language.

2. $\frac{a-x}{4}+\frac{c+d}{5}=\frac{m}{y}.$

3. $\frac{m-n}{6}+c=ab.$

4. $\frac{x+y}{3}-m=\frac{a}{b}.$

5. $ab+\frac{x}{m}=3(a-x).$

6. $(a+b)m+\frac{c}{d}=x-y.$

7. $\frac{a-b}{2}+\frac{c+d}{3}=abc.$

8. $(m+n)x-\frac{a+b}{5}=my-a.$

9. $\frac{a-b}{n}+(x-y)c=\overline{a+b+c}\times d.$

10. $(a+x)(b-y)\cdot\cdot\frac{c}{4}=(a-c)m.$

11. $\frac{a-b}{c}+\frac{x}{m+n}=\frac{abc}{m-y}.$

12. $\frac{a+x}{amn}-\frac{c+d}{xy}=(a+b)(a-b)$.

13. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{x-5}{7}$.

20. Write the following in algebraic language:

Ex. 1. The sum of a and b, multiplied by m, diminished by the quotient of c divided by the difference of m and n, is equal to the product of b, c and d diminished by 7.

Ans. $(a+b)m-\frac{c}{m-n}=bcd-7$.

2. One-fourth of x, diminished by one-fifth of x and increased by one-sixth of x, is equal to 3 times x divided by 8.

Ans. $\frac{x}{4}-\frac{x}{5}+\frac{x}{6}=\frac{3x}{8}$.

3. One-half of y, increased by 4 times y and diminished by one-third of y, is equal to 5 times the difference of y and 4. *Ans.* ——.

4. The sum of m and n, divided by 5, is equal to the difference of a and b multiplied by the sum of x and y. *Ans.* ——.

5. The product of a, b and c, divided by 3, is equal to the sum of a, b and c multiplied by 9. *Ans.* ——.

6. The quotient of m divided by n, increased by the product of c and d, is equal to 3 times the sum of x and y diminished by 10. *Ans.* ——.

7. The product of a and b, diminished by the product of c and d, is equal to the quotient of the sum of a, b and c divided by the difference of x and y. *Ans.* ——.

8. The difference of m and n, divided by b, is equal to the product of a, b and c, diminished by 5 times x increased by the quotient of c divided by d. *Ans.* ——.

9. The product of a and b, divided by c, increased by 4 times n, is equal to the quotient of the sum of h and m, divided by their difference. *Ans.* ——.

10. The sum of a, b and c, multiplied by the difference of m and n, increased by the quotient of x, divided by the sum of c and d, is equal to 3 times y, diminished by 5 times the product of a, c and h. *Ans.* ——.

11. Three times x, diminished by one-half of m and increased by 4 times the product of b, c and d, is equal to the sum of m, n and p, multiplied by the difference of a and b and diminished by one-fourth of h. *Ans.* ——.

12. Four times a, subtracted from 5 times b, increased by the quotient of 3 times c, divided by 4 times d, is equal to the sum of c and d, multiplied by their difference.

Ans. ——.

13. The sum of a and b, multiplied by their difference and divided by their product, is equal to one-third of the sum of m and n, diminished by 10 times y. *Ans.* ——.

14. One-fifth of the product of a, b and c, diminished by 4 times their sum, is equal to 5 times the difference of m and h, divided by $8y$. *Ans.* ——.

21. Still further to enable the student to become expert in the use of algebraic symbols, we will give some examples in algebraic language in which numbers are to be substituted for the letters, and the numerical values are to be written out.

In the examples that follow it will be understood that $a=8$, $b=4$, $c=3$, $d=6$, $m=10$ and $n=2$.

Ex. 1. $\frac{a+b}{c}+\frac{m-n}{b}+3c=$ what?

Substituting for the letters their respective values, we have

$$\frac{8+4}{3}+\frac{10-2}{4}+3\times3=\frac{12}{3}+\frac{8}{4}+9=4+2+9=15. \qquad \textit{Ans. } 15.$$

2. $2a+3b-5c+4n=$ what? *Ans.* 21.

3. $3a+ab-\frac{m}{n}=$ what? *Ans.* 51.

4. $\frac{b+d}{2}-\frac{m+n}{6}+bc=$ what? *Ans.* 15.

5. $(a+b)n-\frac{a}{b}+3mn=$ what? *Ans.* 82.

6. $\frac{m}{n}+\frac{m+a}{c}-\frac{3a}{2b}=$ what? *Ans.* 8.

7. $\frac{abc}{4}-2(a+b)+\frac{a}{n}=$ what? *Ans.* 4.

8. $\frac{m+n}{3}+\frac{a-b}{2}+3(a+b-c)=$ what? *Ans.* 33.

9. $(a+b)n-\frac{m}{d-b}+abn=$ what? *Ans.* 83.

10. $(a+b)(m-n)-\frac{d}{c}+3an=$ what? *Ans.* 142.

22. A *coefficient*[*] is a symbol prefixed to a quantity, to show how many times the quantity is to be *taken*. Thus, $5a$ shows that a is to be *taken* 5 times; it is therefore equivalent to $a+a+a+a+a$, 5 being the coefficient; $4x$ is equivalent to $x+x+x+x$, 4 being the coefficient; $3ab$ is equivalent to $ab+ab+ab$, 3 being the coefficient. When no coefficient is written, 1 is always understood. Thus, x and $1x$ express the same thing; $y=1y$, $am=1am$.

A coefficient may be a fraction. Thus, in the expressions $\frac{3}{4}x$, $\frac{5}{6}y$ and $\frac{7}{8}b$, $\frac{3}{4}$, $\frac{5}{6}$ and $\frac{7}{8}$ are coefficients.

A coefficient may also be a *letter*. Thus, in the expressions ax, bx and cx, a, b and c are coefficients of x. In $9abx$

---

[*] *Coefficient*, from the Latin *coefficio, to make together*—*i. e.* a co-factor; as it *makes* a product with the aid of another factor.

9 may be regarded as the coefficient of abx, or $9a$ as the coefficient of bx, or $9ab$ as the coefficient of x.

Ex. 1. In $7bx$, what is the coefficient of x? of bx?

2. In $4abz$, what is the coefficient of z? of bz? of abz?

3. In $7mnx$, what is the coefficient of x? of nx? of mnx?

4. In $5aby$, what is the coefficient of y? of by? of aby?

23. An *exponent*,* or *index*, is a symbol which is employed to denote that a quantity is to be *used as a factor* a certain number of times. Thus, instead of $b \times b \times b \times b$, or $bbbb$, we simply write b^4, where ^{4} is the exponent or index of b; instead of $yyyyy$, we write y^5, where ^{5} is the exponent of y.

The exponent may be a *letter* as well as a numeral. Thus, x^n shows that x is to be used as a factor as many times as there are units in ^{n}.

An exponent, therefore, is simply a small figure or letter written on the right of a quantity, a little above it, to show how many times the quantity is used as a factor.

Ex. 1. Write $a \times a \times a \times a \times a$ with an exponent. *Ans.* a^5.

2. Write $ccccccc$ with an exponent. *Ans.* ——.

3. Write $xxxxxxxx$ with an exponent. *Ans.* ——.

4. In what two ways can I write c taken as a factor 6 times? *Ans.* $cccccc$, or c^6.

5. How can you express y taken as a factor 7 times? *Ans.* ——.

6. Express m taken as a factor 8 times. *Ans.* ——.

7. Express a^5 without an exponent. *Ans.* $aaaaa$.

8. Express y^4 without an exponent. *Ans.* ——.

---

* *Exponent*, from the Latin *expono*, meaning *to show*, because it shows how many times the quantity is taken as a factor.

9. Express d^5 without an exponent. *Ans.* ——.

10. Write $a^3b^2y^4$ without exponents. *Ans.* $aaabbyyyy$.

11. Express $b^3y^5x^4$ without exponents. *Ans.* ——.

12. Express $bbaabccbbc$ with exponents. *Ans.* $a^2b^5c^3$.

13. Write $bbccxxbcxxc$ with exponents. *Ans.* ——.

14. Write $(a+b)(a+b)(a+b)$ with an exponent. *Ans.* $(a+b)^3$.

15. Write $(a+x)(a+x)(a+x)(a+x)$ with an exponent. *Ans.* ——.

16. Write $4\times4\times4\times4\times4$ with an exponent. *Ans.* 4^5.

17. Write $5\times5\times5\times5$ with an exponent. *Ans.* ——.

NOTE.—A coefficient shows how many times a quantity is *added* to itself; an exponent shows how many times a quantity is *multiplied* into itself. Thus, $4a=a+a+a+a$, but $a^4=aaaa$ or $a\times a\times a\times a$. If $a=3$, then $4a=4\times3=12$, but $a^4=a\times a\times a\times a=3\times3\times3\times3=81$.

Ex. 1. What is the difference between $5x$ and x^5?

2. Show the difference between $6a$ and a^6.

3. What is the difference between $7y$ and y^7?

4. Show the difference between $3x$ and x^3.

24. The successive products formed by multiplying a quantity into itself are called POWERS.

25. Powers are called *first, second, third, fourth,* etc., according to the number of times that the original quantity is used as a factor.

26. The first power of a quantity is the quantity itself. The first power is also called the ROOT of the other powers.

Thus, $a\times a=aa$ is the *second* power of a;
$a\times a\times a=aaa$ is the third power of a;
$a\times a\times a\times a=aaaa$ is the fourth power of a, etc.

Powers are usually indicated by means of exponents.

Thus, $aa = a^2$, where 2 is the exponent of the power.
$aaa = a^3$, where 3 is the exponent of the power.
$aaaa = a^4$, where 4 is the exponent of the power.
$3 \times 3 = 3^2 = 9.$
$3 \times 3 \times 3 = 3^3 = 27.$
$3 \times 3 \times 3 \times 3 = 3^4 = 81.$

When no exponent is expressed, 1 is always understood; thus, a and a^1 signify the same thing; $x = x^1$, $y = y^1$.

Ex. 1. If $a = 3$, what is the numerical value of a^3?
Analysis. $a^3 = a \times a \times a = 3 \times 3 \times 3 = 27.$ *Ans.* 27.

2. If $x = 5$, what is the value of x^2? *Ans.* 25.
3. If $y = 4$, what is the value of y^4? *Ans.* 256.
4. If $b = 2$, what is the value of b^5? *Ans.* 32.
5. What is the value of n^4, if $n = 3$? *Ans.* 81.
6. What is the value of b^6, if $b = 2$? *Ans.* 64.
7. If $y = 6$, what is the value of y^3? *Ans.* 216.
8. What is the value of a^4b^2 when $a = 2$ and $b = 3$? *Ans.* 144.
9. What is the value of b^2n^4 when $b = 6$ and $n = 2$? *Ans.* 576.
10. What is the value of $a^2b^2c^3$ when $a = 3$, $b = 5$ and $c = 2$? *Ans.* 1800.
11. What is the value of $a^3b^2n^2$, when $a = 2$, $b = 3$ and $n = 4$? *Ans.* 1152.

27. A ROOT of a quantity is a factor which, being multiplied into itself a certain number of times, will produce that quantity; thus, 5 is the *square* root of 25, because $5 \times 5 = 25$; 3 is the *cube* root of 27, because $3 \times 3 \times 3 = 27$; a is the square root of a^2, because $a \times a = a^2$; x is the *cube* root of x^3, because $x \times x \times x = x^3$.

28. The symbol $\sqrt{\ }$, called the *radical** sign, is used to denote the square root of a quantity; as $\sqrt{49}=7$, $\sqrt{81}=9$. The cube root is indicated by the figure 3 written in the opening of the radical sign; thus, $\sqrt[3]{27}=3$, $\sqrt[3]{a^3}=a$.

---

* *Radical*, from the Latin *radix*, meaning *root*.

$\sqrt[4]{x}$ denotes the *fourth* root of x.

$\sqrt[5]{y}$ indicates the *fifth* root of y.

$\sqrt[n]{a}$ signifies the nth root of a, n representing any number whatever.

Ex. 1. What is the value of $\sqrt{a}$, if $a=64$? *Ans.* 8.

2. What is the value of $\sqrt[3]{b}$, if $b=64$? *Ans.* ——.

3. What is the value of $\sqrt{x}$, if $x=100$? *Ans.* ——.

4. If $y=81$, what is the value of $\sqrt[4]{y}$? *Ans.* 3.

5. If $a=125$, what is the value of $\sqrt[3]{a}$? *Ans.* ——.

6. What is the value of $\sqrt{x}$ when $x=400$? *Ans.* ——.

7. If $b=216$, what is the value of $\sqrt[3]{b}$? *Ans.* ——.

8. If $a=9$, what is the value of $4\sqrt{a}$? *Ans.* 12.

NOTE.—4 is the coefficient of $\sqrt{a}$, and the expression is read, "4 times the square root of a."

9. If $b=16$, how much is $5\sqrt{b}$? *Ans.* 20.

10. If $a=27$, how much is $6\sqrt[3]{a}$? *Ans.* 18.

11. If $c=49$, what is the value of $4\sqrt{c}$? *Ans.* 28.

12. If $b=64$, how much is $7\sqrt[3]{b}$? *Ans.* 28.

13. If $d=16$, what is the value of $9\sqrt[4]{d}$? *Ans.* 18.

14. What is the value of $5\sqrt[3]{a}$, if $a=8$? *Ans.* 10.

15. If $b=36$, what is the value of $7\sqrt{b}$? *Ans.* 42.

29. Quantities connected by the signs + and − are called TERMS.

30. A quantity consisting of *one term* is called a MONOMIAL. Thus, x, ab and $4ay$ are monomials. A monomial is frequently called a *simple* quantity.

31. A quantity consisting of *more* than one term is called a POLYNOMIAL, or *compound quantity.*

32. A polynomial of *two* terms is called a BINOMIAL. Thus, $(a+b)$, $(a-b)$, $(m+n)$, $(a+x)$, etc. are binomials. $(a+b)$ is called a *positive* binomial; $(a-b)$ is called a *residual* * binomial.

33. A polynomial of *three* terms is called a TRINOMIAL.

34. SIMILAR or LIKE terms are those which consist of the *same letters with the same exponents.* Thus, $3ax$, $5ax$ and $-8ax$ are similar terms; so are $5xy^2$ and $3xy^2$. But $5ax$ and $5by$ are *unlike* terms, because the letters are not the same; $5by$ and $5b^2y^2$ are *unlike,* because, while the *letters* are the same, the *exponents* are different.

35. The RECIPROCAL of a quantity is the quotient arising from dividing a unit by that quantity. Thus, the reciprocal of 5 is $\frac{1}{5}$; of 7 is $\frac{1}{7}$; of a is $\frac{1}{a}$; of $(a+b)$ is $\frac{1}{a+b}$; of $\frac{1}{2}$ is $\frac{1}{\frac{1}{2}}$, or 2; of $\frac{2}{3}$ is $\frac{1}{\frac{2}{3}}$, or $\frac{3}{2}$; of $\frac{a}{b}$ is $\frac{1}{\frac{a}{b}}$, or $\frac{b}{a}$. Hence, the *reciprocal* of *any* quantity is that quantity inverted.

36. As it is a matter of the highest importance that the student should be perfectly familiar with the notation and uses of algebraic language, we append a series of examples to serve as a recapitulation of the general principles enunciated in the foregoing pages. It will be understood that in all the examples $a=5$, $b=6$, $m=3$ and $n=4$.

Ex. 1. What is the value of $am+n$? *Ans.* 19.

2. What is the value of $3amn-2n$? *Ans.* 172.

3. What is the value of $(a+m)n-5a$? *Ans.* 7.

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* *Residual,* from the Latin *residuum,* meaning *remainder,* because it expresses what is *left* after a subtraction.

4. What is the value of $(a+m+n)a$? *Ans.* 60.

5. What is the value of $(a+m)(b+n)$? *Ans.* 80.

6. What is the value of $(a+b+m)n$? *Ans.* 56.

7. What is the value of $\frac{a+m}{2}-\frac{b+n}{5}+2a$? *Ans.* 12.

8. What is the value of $5abm-4n$? *Ans.* 434.

9. What is the value of $3a+2bm-2an$? *Ans.* 11.

10. What is the value of a^2+b^2-2mn? *Ans.* 37.

11. What is the value of $a^2+b^2-m^3+n$? *Ans.* 38.

12. What is the value of $\frac{m^2+n^2}{5}+abm$? *Ans.* 95.

13. What is the value of $\frac{b^2-m^3}{3}+\frac{b+n}{2}+m^2$? *Ans.* 17.

14. What is the value of $\frac{2m+3n}{b}+(a+b)m$? *Ans.* 36.

15. What is the value of $\sqrt{n}+\sqrt{3m}+\sqrt{5a}$? *Ans.* 10.

16. What is the value of $ab-\sqrt[3]{2n}+\frac{3a}{5}$? *Ans.* 31.

17. What is the value of $\frac{n}{2}+\frac{m}{3}-\frac{n}{4}+2mn$? *Ans.* 26.

18. What is the value of $3\sqrt{n}+(a+b)(m+n)$? *Ans.* 83.

19. What is the value of $\sqrt{4n}+\sqrt{12m}-\sqrt{n}$? *Ans.* 8.

20. What is the value of a^2b+ab^2-7mn? *Ans.* 246.

21. What is the value of $a^2b^2-3ab+5am$? *Ans.* 885.

22. What is the value of $\sqrt{a+n}+m^2n^2$? *Ans.* 147.

23. What is the value of $\overline{a+b+m}\times n+b^3$? *Ans.* 272.

24. If $4x=b^2+n^2+2mn$, what is the value of x?

Analysis. By substituting the values of the letters, we find $4x=76$; then, if $4x=76$, $1x$ will be equal to $\frac{1}{4}$ of $76=\frac{76}{4}=19$ *Ans.*

25. If $5x=(a+m)(b-n)+m^2$, what is the value of x?

Ans. $x=5$.

26. If $6y = b^2+m^2+(a+b)m$, what is the value of y?

Ans. $y = 13$.

The use of signs and letters may be clearly shown by a few simple problems:

Ex. 1. A merchant has two pieces of muslin which together measure 72 yards, and the second piece contains twice as many yards as the first; how many yards are in each piece?

In this problem *two unknown* quantities are to be sought after—viz. the number of yards in each of the pieces.

Let x = No. yards in the first piece, then $2x$ will equal the No. yards in the second piece, and as the two pieces together measure 72 yards, we have

$$x+2x = 72 \text{ yards.}$$

But $x+2x = 3x$; therefore, $3x = 72$ yards; and if $3x = 72$ yards, then $1x$ will equal $\frac{1}{3}$ of 72 yards $= \frac{72 \text{ yards}}{3} = 24$ yards in the *first* piece; and as the second piece contains *twice* as many yards as the first, the second will contain $2 \times 24 = 48$ yards. *Ans*, 1st, 24 yds.; 2d, 48 yds.

In this simple example we see how it is that the *unknown* quantity may be represented by a letter, and may enter into the statement of the problem just as though it were known. One of the distinguishing particulars in which Algebra differs from Arithmetic is the fact that Algebra introduces *unknown* quantities into calculation, and operates with them according to the rules of Arithmetic, just as though they were already *known*.

Ex. 2. A milliner bought three bonnets for $18; the second cost twice as much as the first, and the third three times as much as the first; what was the cost of each?

Let x = the cost of the first,
then $2x$ = " " " second,
and $3x$ = " " " third.

And as the three together cost 18 dollars, we have

$$x+2x+3x = 18 \text{ dollars.}$$

But *once* x+twice x+3 times x is equal to *six* times x; hence,

$$6x=18 \text{ dollars}.$$

And if $6x=18$ dollars, then $1x$ or x will equal $\frac{1}{6}$ of 18 dollars $=\frac{18}{6}$ dollars $=3$ dollars, the cost of the first bonnet; and as the second cost *twice* as much as the first, the second cost $2\times3=6$ dollars; and as the third cost *three* times as much, the third cost $3\times3=9$ dollars.

Ans. 1st, \$3; 2d, \$6; 3d, \$9.

3. A farmer has 4 flocks of sheep, the whole amounting to 180. The second contains twice as many as the first, the third three times as many as the first, and the fourth four times as many as the first; how many sheep in each flock?

Ans. 1st, 18; 2d, 36; 3d, 54; 4th, 72.

4. A boy bought 5 school-books, the whole cost amounting to \$2.25. The second cost twice as much as the first, the third three times, the fourth four times, and the fifth five times as much as the first; what was the cost of each book?

Ans. 1st, 15 cts.; 2d, 30 cts.; 3d, 45 cts.; 4th, 60 cts.; 5th, 75 cts.

5. A man bought a number of calves, hogs and sheep, the whole numbering 84. There were twice as many hogs as calves, and twice as many sheep as hogs; how many of each did he buy?

Let $x=$No. of calves; and as the hogs were twice as many as the calves, then $2x=$No. of hogs; and as the sheep were twice the number of hogs, then $4x=$No. of sheep. Hence, we have

$$x+2x+4x=84.$$

Uniting terms,

$$\begin{aligned} 7x&=84\\ x&=\tfrac{84}{7}=12=\text{No. calves},\\ 2x&=24=\text{No. hogs},\\ 4x&=48=\text{No. sheep}. \end{aligned}$$

6. A hunter went out gunning and killed a number of birds, amounting to 42 in all. There were twice as many robins as doves, and twice as many larks as robins; how many of each sort did he kill?

Ans. 6 doves; 12 robins; 24 larks.

7. A man travelled for 3 days on a railroad, the whole distance being 405 miles. The second day he went twice as far as on the first, and on the third *three* times as far as on the second; how far did he travel each day?

Ans. 1st, 45 miles; 2d, 90 miles; 3d, 270 miles.

8. A commission-merchant bought cotton for three weeks, the whole amounting to 840 bales. The second week he bought three times as many bales as in the first week, and the third week twice as many bales as during the second; how many bales did he buy each week?

Ans. 1st, 84 bales; 2d, 252 bales; 3d, 504 bales.

9. A farmer has barley, rye, wheat and oats, the whole amounting to 816 bushels. There are three times as many bushels of rye as barley, twice as many bushels of wheat as rye, and four times as many bushels of oats as wheat; how many bushels of each kind of grain has he?

Ans. Barley, 24 bu.; rye, 72 bu.; wheat, 144 bu.; oats, 576 bu.

10. A farmer has cows, goats, hogs and sheep, the whole number amounting to 205. There are four times as many goats as cows, 3 times as many hogs as goats, and twice as many sheep as hogs; how many of each kind has he?

Ans. 5 cows; 20 goats; 60 hogs; 120 sheep.

11. A bag of silver contains 105 pieces of money, consisting of halves, quarters and dimes. There are five times as many quarters as halves, and three times as many dimes as quarters; how many of each kind are there?

Ans. 5 halves; 25 quarters; 75 dimes.

12. A farm containing 300 acres is planted in wheat, corn and cotton. There are six times as many acres of corn as of wheat, and three times as many acres of cotton as of corn; how many acres of each?

Ans. Wheat, 12 acres; corn, 72 acres; cotton, 216 acres.

13. An orchard contains 845 trees, consisting of quince,

pear, cherry, apple and peach. There are three times as many pear trees as quince, five times as many cherry as pear, twice as many apple as cherry, and four times as many peach as apple; how many of each sort are there?

Ans. 5 quince; 15 pear; 75 cherry; 150 apple; 600 peach.

POSITIVE AND NEGATIVE QUANTITIES.

37. In Arithmetic all the quantities that enter into calculation are *positive:* the student of Arithmetic knows no other kind. As soon, however, as he enters upon the study of Algebra he is met by another class of quantities, termed *negative.* The negative quantities frequently give the beginner a considerable amount of trouble, in consequence of his not obtaining a clear comprehension of their nature. As a *positive* quantity is *real,* the student is apt to suppose that a *negative* quantity must be *imaginary*—that as a positive quantity is a *something,* a negative quantity must be a sort of mathematical *nothing.* This notion is very erroneous. The principal point of difference between a positive and a negative quantity is, that a positive quantity is one which is always to be *added;* a negative quantity, on the contrary, is one always to be *subtracted.* Thus, if a merchant is conducting business his *gains* are *positive,* because they *add to,* or *increase,* his capital, while his *losses* are *negative,* because they *subtract from,* or *diminish,* his capital. Thus, if he sets out with a capital of 5000 dollars, and for the first month *gains* 200 dollars, during the second *loses* 150 dollars, then gains 100, then loses 50 dollars, his account will stand as follows:

$$5000+200-150+100-50=5100 \text{ dollars.}$$

The result shows, after aggregating his gains and his losses, that he has a *net* gain of 100 dollars, his capital now being 100 dollars greater than that with which he commenced.

Suppose him to start with the same capital as before, and suppose that at first he *loses* 200 dollars, then *gains* 150 dollars, then loses 100, then gains 50; his account will then stand,

$$5000-200+150-100+50=4900 \text{ dollars.}$$

The result indicates an aggregate *loss* of 100 dollars, his capital now being 100 dollars less than that with which he began. Yet the *loss* in the one case is as much a *reality* as the *gain* in the other.

Again: Suppose a ship to start from the equator, and sail alternately northward and southward. From the relation which exists between a positive and a negative quantity, if the northward motion is *positive*, the southward motion must be *negative*. If the ship sails now 20° N., then 15° S., then 15° N., then 10° S., her position would then be as follows:

$$+20^\circ-15^\circ+15^\circ-10^\circ=+10^\circ.$$

The result, $+10^\circ$, shows that the ship is 10° north of the equator.

Again: Suppose the ship to start again from the equator, and to sail 25° N., then 20° S., then 30° N., then 40° S.; her position would then be

$$+25^\circ-20^\circ+30^\circ-40^\circ=-5^\circ.$$

The result, -5°, denotes that her motion southward has exceeded her motion northward, and that she is now 5° S. of the equator.

In estimating the value of a man's estate, money, houses, land, etc. are regarded as *positive*, because their tendency is to *increase* the value of the estate; but *debts*, or money due to others, must be regarded as *negative*, because their tendency is to *diminish* the estate.

Thus, if a man's property in possession amounts to 5000 dollars, and his debts to 1500 dollars, his estate will then stand,

$$+5000-1500=+3500,$$

showing an aggregate net value of 3500 dollars.

On the contrary, suppose his property is 5000 dollars, and his debts 6500, then his estate will be represented by

$$+5000-6500=-1500 \text{ dollars.}$$

The result, -1500 dollars, indicates that his debts have exhausted his entire property, and left 1500 dollars still unpaid.

The signs + and −, as representatives of the words *positive* and *negative,* indicate an *opposition* of some sort, so that the quantities to which these signs are prefixed increase in the one case and decrease in the other the quantities with which they are connected. Thus, if a forward motion is +, then a backward motion is −; if eastward progress is +, then westward progress is −; in a merchant's account money that *comes in* must be regarded as +, because it *increases* the stock, while money that *goes out* is −, because it *decreases* the stock.

A negative quantity is just as much a reality as a positive quantity, the only difference being that in the case of a positive quantity the effect is to *increase* the quantities with which it is connected, and in that of a *negative* quantity the effect is to *diminish* the quantities with which it is connected.

It therefore follows, as a natural consequence, that *to add a negative quantity has exactly the same effect as to subtract a positive quantity of the same numerical value.* If a man owns 5000 dollars' worth of property, and if by endorsing a note for 500 dollars he is afterward required to pay the same, a *debt* of 500 dollars is virtually *added* to his estate, making his property only 4500 dollars. This would actually be the same, in effect, as to take the positive sum of 500 dollars from his estate. This would be represented, algebraically, thus:

$$5000+(-500)=5000-500=4500.$$

It also follows, as a necessary consequence, that *to subtract a negative quantity has exactly the same effect as to add an equal positive quantity.*

Thus, if a man is in possession of 5000 dollars' worth of property, and at the same time owes a debt of 500 dollars, his estate is really worth only 4500 dollars. Now, suppose we *take away* this debt; *i. e. cancel it* or *annul it;* we make his estate 5000 dollars, or, if we allow the debt to remain and *add* 500 dollars to his estate, we produce exactly the same result; *i. e.* we make his estate 5000 dollars.

Another important truth is, *that the algebraic difference between a positive quantity and a negative quantity is always* GREATER *than either of the quantities.* Thus, on a thermometer the difference between 20° above zero and 10° *below* zero (*i. e.* the difference between $+20°$ and $-10°$) is $+30°$. The difference between 30° N. latitude and 20° S. latitude (*i. e.* between $+30°$ and $-20°$) is $+50°$.

One man has 500 dollars in his pocket (*i. e.* he has $+500$ dollars). Another man not only has no money, but is 500 dollars in *debt* (*i. e.* his money is represented by -500 dollars). The difference in the money of the two men is evidently 1000 dollars, because it would be necessary to add to the man who is in debt 500 dollars in order to cancel his debt, and then he would be worth simply *nothing;* and it would then require 500 dollars more in order to make him worth 500 dollars.

We have dwelt at some length on this subject, because it is a matter of the greatest importance to the beginner in Algebra that he should obtain a clear comprehension of the difference between positive and negative quantities

Indeed, there can be no profitable or satisfactory progress without a thorough knowledge of this subject.

AXIOMS.

38. An Axiom is a proposition the truth of which is so evident at first sight that no process of reasoning can make it clearer.

Investigations in Algebra are carried on mainly by means of *equations* and *proportions*, and the processes connected with these are founded upon a few simple principles or axioms, the most important of which will now be given:

Axiom 1. If the same quantity or equal quantities be *added* to equal quantities, their *sums* will be equal.

2. If the same quantity or equal quantities be *subtracted* from equal quantities, the *remainders* will be equal.

3. If equal quantities be multiplied by the same or by equal quantities, the *products* will be equal.

4. If equal quantities be divided by the same or by equal quantities, the *quotients* will be equal.

5. If the same quantity be both *added to* and *subtracted from* another, the value of the latter will not be altered.

6. If a quantity be both *multiplied* and *divided* by another, the value of the former will not be altered.

7. If to unequal quantities equals be added, the greater will give the greater sum.

8. If from unequal quantities equals be subtracted, the greater will give the greater remainder.

9. If unequal quantities be multiplied by equals, the greater will give the greater product.

10. If unequal quantities be divided by equals, the greater will give the greater quotient.

11. Quantities which are equal to the same quantity are equal to each other.

12. The whole of a quantity is greater than a part.

13. The whole of a thing is equal to the sum of all its parts.

SECTION II.

ADDITION.

39. Addition *in Algebra is the connecting of two or more quantities by their proper signs, so as to represent their aggregate value.*

Thus, m added to n is $m+n$, and y added to the sum of m and n is $m+n+y$.

Again: $4x$ added to $6x$ is $10x$; for, if the symbol x be taken to represent any quantity or thing which is the subject of calculation, as dollars, pounds, etc., it is obvious that $4x$ will represent *four* times that thing, and $6x$ will represent *six* times the same thing, whatever the thing may be; and consequently *four* times any quantity + *six* times the same quantity must be *ten* times the quantity; *i. e.* $4x+6x=10x$.

EXAMPLES.

	(1.)	(2.)	(3.)	(4.)	(5.)
	$4x$	$5a$	$4ax$	$5a^2$	$3axy$
	$2x$	$4a$	$3ax$	$4a^2$	$4axy$
	$3x$	$3a$	$2ax$	$3a^2$	axy
	x	a	$5ax$	a^2	$9axy$
	$7x$	$7a$	$6ax$	$6a^2$	$8axy$
Ans.	$17x$	$20a$	$20ax$	$19a^2$	$25axy$

6. Add together $3a+4a+7a+5a+4a$. *Ans.* $23a$.
7. Add together $6x+4x+5x+3x+2x$. *Ans.* ——.
8. What is the sum of $3y+5y+7y+9y+5y$? *Ans.* $29y$.
9. What is the sum of $5y+8y+9y+7y+3y$? *Ans.* ——.

10 What is the sum of $3ab+4ab+ab+9ab+6ab$?

Ans. $23ab$.

11. What is the sum of $7b^2+4b^2+b^2+8b^2+6b^2$?

Ans. ——.

12. What is the sum of $6ay+4ay+ay+9ay+5ay$?

Ans. $25ay$.

13. What is the sum of $7bc+3bc+bc+8bc+3bc$?

Ans. ——

14. What is the sum of $4ac+3ac+5ac+9ac+8ac$?

Ans. $29ac$.

15. What is the sum of $3x^2+4x^2+7x^2+x^2+6x^2$?

Ans. ——.

	(16.)	(17.)	(18.)	(19.)
	$3a+2x$	$4x+2y$	$2n+3a^2$	$2ax+3b$
	$4a+6x$	$3x+3y$	$4n+4a^2$	$4ax+5b$
	$5a+x$	$5x+y$	$n+5a^2$	$7ax+b$
	$a+3x$	$x+6y$	$5n+a^2$	$3ax+4b$
	$7a+4x$	$6x+4y$	$3n+7a^2$	$2ax+7b$
Ans.	$20a+16x$			

	(20.)	(21.)	(22.)	(23.)
	$4x^2+3xy$	$5ab+6y^2$	$3m+4ay$	$4a^3+6ac$
	$2x^2+4xy$	$4ab+4y^2$	$5m+6ay$	$5a^3+4ac$
	$6x^2+xy$	$2ab+y^2$	$2m+ay$	$7a^3+ac$
	$7x^2+8xy$	$3ab+7y^2$	$4m+5ay$	$4a^3+7ac$
Ans.	$19x^2+16xy$			

	(24.)	(25.)	(26.)	(27.)
	$2an+3b^3$	$5bc+6x^4$	$3a+4b$	$5c+6x^3$
	$5an+4b^3$	$4bc+2x^4$	$2a+3b$	$7c+9x^3$
	$3an+b^3$	$8bc+x^4$	$9a+7b$	$8c+6x^3$
	$6an+8b^3$	$7bc+6x^4$	$8a+9b$	$9c+8x^3$
Ans.	$16an+16b^3$			

40. The same principle of addition applies when the signs of all the quantities are $-$. In adding any number of *negative* quantities the result must evidently be a *negative* sum. Merely adding quantities together cannot alter the character of the quantities. Pounds added to pounds will give pounds as the result; *debts* added together will evidently still be *debts*. Hence, the sum of $-4x$ and $-6x$ must obviously be $-10x$; also, $-7b$ and $-8b$ must be $-15b$. Hitherto, the student has been told that a *negative* quantity is one that is always to be *subtracted*, and he may suppose that we have violated this general principle in *adding* $-7b$ and $-8b$, making the sum $-15b$. But the operation is perfectly consistent. It is still true that $-7b$ and $-8b$ are both to be *subtracted*, not from each other, but from some other quantity that is understood. For instance, $-7b$ may represent a *debt* of 7 dollars which a man owes to one creditor, and $-8b$ may express a *debt* of 8 dollars due to another creditor. If the man is worth 1000 dollars, the effect on his property will be the same whether we *subtract* each debt separately or unite the two debts, making 15 dollars, and subtract the entire sum of 15 dollars at once.

EXAMPLES.

	(1.)	(2.)	(3.)	(4.)	(5.)
	$-4a$	$-3x^2$	$-7ab$	$-5a^3$	$-6n$
	$-5a$	$-5x^2$	$-5ab$	$-4a^3$	$-7n$
	$-7a$	$-7x^2$	$-6ab$	$-8a^3$	$-4n$
	$-6a$	$-8x^2$	$-9ab$	$-7a^3$	$-8n$
	$-4a$	$-2x^2$	$-8ab$	$-5a^3$	$-6n$
Ans.	$-26a$	$-25x^2$	$-35ab$	$-29a^3$	$-31n$

6. What is the sum of $-9x$, $-4x$, $-8x$, $-7x$ and $-8x$?
Ans. $-36x$?

7. What is the sum of $-4ay$, $-7ay$, $-6ay$ and $-9ay$?
Ans. ——.

8. Add together $-4a^2$, $-5a^2$, $-6a^2$ and $-9a^2$.

Ans. $-24a^2$.

41. The operations in addition may be classed under three different cases.

CASE I.

When the Quantities are Similar and have Like Signs.

To meet this case we have from the foregoing illustrations the following

RULE.—*Add the coefficients, annex the common letter or letters and prefix the common sign.*

EXAMPLES.

	(1.)	(2.)	(3.)	(4.)
	$3a - 2x$	$3x - 4y$	$4n - 5a$	$7b - 4m$
	$5a - 4x$	$5x - 7y$	$6n - 7a$	$6b - 7m$
	$6a - 7x$	$4x - 5y$	$5n - 3a$	$4b - 5m$
	$9a - 8x$	$7x - 8y$	$8n - 9a$	$9b - 8m$
	$5a - 7x$	$3x - 5y$	$6n - 7a$	$7b - 6m$
Ans.	$28a - 28x$			

	(5.)	(6.)	(7.)	(8.)
	$6ax - 4b^2$	$7ay - 3x^3$	$5an - 4c^2$	$3a^3 - 6bc$
	$9ax - 6b^2$	$5ay - 5x^3$	$6an - 3c^2$	$4a^3 - 7bc$
	$7ax - 3b^2$	$6ay - 7x^3$	$4an - c^2$	$7a^3 - bc$
	$5ax - b^2$	$5ay - 4x^3$	$7an - 9c^2$	$5a^3 - 4bc$
	$6ax - 9b^2$	$7ay - 9x^3$	$5an - 7c^2$	$8a^3 - 9bc$
Ans.	$33ax - 23b^2$			

	(9.)	(10.)	(11.)	(12.)
	$5xy - 4b$	$4b^3 - 6ab$	$5x^2 - 4an$	$6ax - 4x^3$
	$4xy - 3b$	$6b^3 - 5ab$	$6x^2 - 6an$	$9ax - 8x^3$
	$7xy - b$	$7b^3 - 3ab$	$8x^2 - 9an$	$8ax - 6x^3$
	$6xy - 9b$	$9b^3 - 8ab$	$9x^2 - 8an$	$7ax - 5x^3$
	$5xy - 8b$	$6b^3 - 7ab$	$7x^2 - 6an$	$4ax - 7x^3$
Ans.	$27xy - 25b$			

13. Add $2x+4a$, $3x+2a$, $5x+7a$, $8x+6a$ and $4x+5a$.
Ans. $22x+24a$.

14. Add $3a^2-4xy$, $5a^2-3xy$, $6a^2-2xy$, $7a^2-9xy$ and $8a^2-5xy$.
Ans. ——.

15. Add $3a-2bc+4x$, $6a-5bc+6x$, $8a-5bc+3x$ and $9a-6bc+7x$.
Ans. $26a-18bc+20x$.

16. Add $5x^3+3x^2-7x$, $6x^3+5x^2-8x$, $4x^3+6x^2-5x$ and $7x^3+8x^2-9x$.
Ans. ——.

In the same manner, any similar quantities whatever may be added by taking the algebraic sum of their coefficients.

(17.)	(18.)	(19.)
$5(a+b)-\ \ 3\sqrt{x}$	$7(a-b)-\ \ 4\sqrt{x+y}$	$3(a+x)-\ \ 6\sqrt[3]{a}$
$6(a+b)-\ \ 5\sqrt{x}$	$6(a-b)-\ \ 5\sqrt{x+y}$	$5(a+x)-\ \ 7\sqrt[3]{a}$
$7(a+b)-\ \ 9\sqrt{x}$	$9(a-b)-\ \ 8\sqrt{x+y}$	$7(a+x)-\ \ 9\sqrt[3]{a}$
$5(a+b)-\ \ 8\sqrt{x}$	$7(a-b)-\ \ 6\sqrt{x+y}$	$9(a+x)-\ \ 5\sqrt[3]{a}$
$23(a+b)-25\sqrt{x}$	$29(a-b)-23\sqrt{x+y}$	$24(a+x)-27\sqrt[3]{a}$

CASE II.

42. When the Quantities are Similar, but have Unlike Signs.

Ex. 1. What is the sum of $5x, -3x, +6x, -2x$?

Analysis. $5x+6x=11x$; and $-3x, -2x=-5x$.
Then $11x-5x=6x$ *Ans.*

Here we unite the + quantities in one term, making $+11x$; and also the − quantities in one term, making $-5x$. Then $11x-5x=6x$.

OPERATION.

$$\begin{array}{r} +5x \\ -3x \\ +6x \\ \underline{-2x} \\ \textit{Ans.}\ +6x \end{array}$$

Here the effect of the *addition* is to *diminish* the numerical result. This, however, is perfectly consistent. Suppose x to represent *one* dollar; then $5x$ will express 5 dollars, and so on. Now, if $+5x$ and $+6x$ represent the *profits* made in trading, then evidently $-3x$ and $-2x$ must represent the *losses.* Our question will then be this: Suppose a man to *gain* 5 dollars, then to *lose* 3 dollars, then to *gain* 6 dollars, and finally to *lose* 2 dollars, how will his account then stand? We should express the result in this way:

$$+5-3+6-2=+6.$$

Adding his *profits,* we find the sum to be 11 dollars; adding his *losses,* we find them to be 5 dollars. Uniting his profits and his losses, we find the aggregate to be $+6$ dollars; *i. e.* his *profits* have exceeded his *losses,* and he has a balance of 6 dollars in his favor.

Ex. 2. What is the sum of $+5b$, $-4b$, $+6b$, $-9b$?

Analysis. $+5b+6b=+11b$; and $-4b$, $-9b=-13b$.
Then, $+11b-13b=-2b$.

As was shown in the preceding article, $+11b$ may be regarded as 11 dollars of *profit,* and $-13b$ may be considered as 13 dollars of *losses,* and uniting 11 dollars of profit with 13 dollars of loss, we see that the losses have exceeded the profits, leaving 2 dollars of *loss* to be made up; which we represent by $-2b$.

OPERATION.

$$\begin{array}{rr} & +5b \\ & -4b \\ & +6b \\ & -9b \\ \hline \textit{Ans.} & -2b \end{array}$$

Hence, we derive the following

RULE.—*Add together all the positive coefficients, and also all those that are negative; subtract the smaller of these sums from the greater, annex the common letter or letters, and prefix the sign of the greater sum.*

NOTE.—When no sign is expressed, + is always understood; and when no numerical coefficient is written, 1 is always understood.

EXAMPLES.

1. What is the sum of $7ax-3ax+4ax-2ax$? *Ans.* $6ax$.
2. What is the sum of $6a^2-4a^2+3a^2-8a^2$? *Ans.* $-3a^2$.
3. What is the sum of $6ay-2ay+8ay-5ay$? *Ans.* $7ay$.
4. What is the sum of $5xy-6xy+4xy-7xy$?
Ans. $-4xy$.

(5.)

$$\begin{array}{lr} & 4a-5b \\ & -2a+7b \\ & 8a-6b \\ & -3a+2b \\ \hline \textit{Ans.} & 7a-2b \end{array}$$

(6.)

$$\begin{array}{r} 6x-4ab \\ -2x+3ab \\ 6x+5ab \\ -4x-2ab \\ \hline \end{array}$$

(7.)

$$\begin{array}{r} 7y-2b^2 \\ -5y+4b^2 \\ 6y+3b^2 \\ -4y-7b^2 \\ \hline \end{array}$$

(8.)

$$\begin{array}{r} 5a-4n \\ -7a+6n \\ 9a-8n \\ -3a+2n \\ \hline \end{array}$$

(9.)

$$\begin{array}{lr} & 5(a+b)+3a^2 \\ & -2(a+b)-7a^2 \\ & 6(a+b)+2a^2 \\ & -3(a+b)-5a^2 \\ \hline \textit{Ans.} & 6(a+b)-7a^2 \end{array}$$

(10.)

$$\begin{array}{r} 3x+5(a-b) \\ -5x+3(a-b) \\ 7x-6(a-b) \\ 4x+3(a-b) \\ \hline \end{array}$$

(11.)

$$\begin{array}{r} 6(x+y)-4x^2 \\ -2(x+y)+3x^2 \\ 5(x+y)-5x^2 \\ -3(x+y)+4x^2 \\ \hline \end{array}$$

(12.)

$$\begin{array}{r} 4x^2+5ab-3a^2 \\ 3x^2-7ab+6a^2 \\ -2x^2+4ab-3a^2 \\ 5x^2-6ab+7a^2 \\ \hline 10x^2-4ab+7a^2 \end{array}$$

(13.)

$$\begin{array}{r} 6x+3y-9ab+5 \\ -3x+5y+7ab-7 \\ 5x-7y+8ab+4 \\ -2x+3y-9ab-3 \\ \hline \end{array}$$

(14.)

$$\begin{array}{r} 7a-5b+3x-4y \\ -3a+7b-4x+2y \\ 9a-8b+9x-8y \\ -4a+3b-5x+7y \\ \hline \end{array}$$

(15.)

$$\begin{array}{lr} & 5y^2-4y+6(a+b)-5a \\ & -6y^2+8y-8(a+b)+7a \\ & 5y^2-9y+7(a+b)-8a \\ & -2y^2+3y-2(a+b)+3a \\ \hline \textit{Ans.} & 2y^2-2y+3(a+b)-3a \end{array}$$

(16.)

$$\begin{array}{r} 5ab-4x^2+3(a-b)+5 \\ -2ab+6x^2-5(a-b)+7 \\ 4ab-7x^2+8(a-b)-9 \\ -3ab+3x^2-4(a-b)+3 \\ \hline \end{array}$$

CASE III.

43. When Some of the Quantities are Alike, and Others are Unlike.

When *all* the quantities are *unlike* they can be added only by writing them in succession, and giving to each its proper sign. Thus, $4ax$, $3y^2$, $-5x^2$ and $7n$ are *unlike* quantities, and their sum is simply $4ax+3y^2-5x^2+7n$.

Ex. 1. Add $3x^2+5y^2-3ab+5$, $5x^2-3y^2+2$ and $2x^2-4y^2-3$.

In this example we unite those quantities that are *alike*, just as in the two preceding cases. One quantity, $-3ab$, had no corresponding *like* quantity, and we simply brought this down, connecting it with the sum of the others by its proper sign.

OPERATION.

$$\begin{array}{l} 3x^2+5y^2+5-3ab \\ 5x^2-3y^2+2 \\ \underline{2x^2-4y^2-3} \\ 10x^2-2y^2+4-3ab. \end{array}$$

In cases of this kind it will be convenient to write similar quantities in the same column, and then proceed according to the following

RULE.—*Add all the like quantities, just as in the two preceding cases, and to their sum annex the unlike quantities in succession, with their proper signs.*

EXAMPLES.

1. What is the sum of $4x+2y-3$, $5n+6x^2+5$ and $3x-5y+2$? *Ans.* $7x+5n-3y+6x^2+4$.

2. Add $5a+4b-6c$, $3a-6b+4c+3d$ and $2a-3b+4c-4x$. *Ans.* $10a-5b+2c+3d-4x$.

3. Add $2x^2+3y^2-5a+5(a+b)$, $5a^2-6y^2+7a-9(a+b)$ and $5x^2+6n-2a+2(a+b)$. *Ans.* $5a^2+7x^2-3y^2-2(a+b)+6n$.

4. Add $6xy-4a^2+3c-6$, $5n+2xy+8a^2+4$ and $4a^2-8xy+2c-5ab$. *Ans.* $8a^2+5c-2+5n-5ab$.

5. Add $4x-5y+3ab-7c^2+4$, $5ab+3c^2-2+6y-2x$, $3y-2ab+3x-7+4xy$ and $5ab-3x+6y+4c^2+5$. *Ans.* $2x+10y+11ab+4xy$.

6. Add $4a^2-5xy+3ab-5x^2+3$, $2xy-5ab+3a^2-5+3x^2$, $6ab-3xy-2a^2-5+2x^2$ and $4x^2-7-5a^2+6xy+4ab+3y^2$. *Ans.* $8ab+4x^2-14+3y^2$.

7. Add $5ax+3b^2+4(a+x)-7ay$, $4b^2-3(a+x)+6ay-8ax$, $5(a+x)-3ay+6ax-5b^2$ and $2ay-2b^2+5ax-3(a+x)+7n$. *Ans.* $8ax+3(a+x)-2ay+7n$.

44. One of the points of difference between Algebra and Arithmetic is seen in the operation of addition. In Arithmetic, the addition of one quantity to another necessarily implies *augmentation* or *increase,* because the addition is always limited to *positive* quantities, these being the only kind known to Arithmetic. But in Algebra the case is different. The algebraic *sum* of a positive quantity and a negative quantity is always numerically *less* than the one or the other of the quantities. Thus, +10 added to −6 will give +4, a quantity *less* than either of the two quantities, numerically considered. And yet this is perfectly consistent. As all those quantities which have the sign + prefixed to them are to be *added,* and those which have the sign − are to be *subtracted,* it follows that those quantities with the + sign *increase,* and those with the − sign *diminish,* the quantities with which they are connected. Adding a *negative* quantity has the same effect as *subtracting* an equal positive quantity. Adding a *debt* of 200 dollars to a man's property is the same, in effect, as to *take away* 200 dollars from his estate. The *algebraic sum* of a positive and a negative quantity is always *less* than the positive quantity.

SECTION III.

SUBTRACTION.

45. SUBTRACTION *is the operation of finding the difference between two quantities.*

The MINUEND* is the quantity from which we are to subtract.

The SUBTRAHEND† is the quantity to be subtracted.

The REMAINDER is that which is left after subtracting.

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* *Minuend,* from the Latin *minuendus,* meaning *to be diminished.*

† *Subtrahend,* from the Latin *subtrahendus,* meaning *to be subtracted.*

46. If 4 is to be *added* to 6, the operation is expressed thus: $6+4=10$. Now, as subtraction is the *reverse* of addition, it follows that if 4 is to be *subtracted* from 6, it will be expressed by $6-4=2$.

Again: If b is to be added to a, we express the operation by $a+b$; if b is to be subtracted from a, we write it thus: $a-b$; from which it appears that to subtract any positive quantity from another we must change the sign of the quantity to be subtracted from + to −.

Again: Suppose we wish to subtract $(c-n)$ from a, the result will be $a-(c-n)$, which is equal to $a-c+n$. This will be evident from considering that if c be taken from a the result will be $a-c$. But it is not c that we are to subtract from a, but a quantity less than c by the units in n; hence, in taking away c we have taken *too much* by the units in n, and therefore the remainder is *too small* by n; and therefore n must be *added to* the result to make it correct, which gives $a-c+n$ as the true remainder.

In the process of subtraction the signs of the subtrahend have been changed; *i. e.* $+c$ has become $-c$, and $-n$ has been changed to $+n$, and the terms were then connected as in addition. This can be more strikingly exhibited by using *numbers*, as in Arithmetic. If 5 were to be subtracted from 10, the difference would be $10-5=5$; now, if $(5-3)$ instead of 5 were to be subtracted from 10, it is obvious that the remainder would be *greater* by 3 than if only 5 were subtracted; *i. e.* $10-(5-3)=10-5+3=8$.

From these illustrations we learn that a *positive* quantity is subtracted by changing its sign to *minus*, and a *negative* quantity by changing its sign to *plus*. For all cases in subtraction we have the following

RULE.—*Conceive the signs of all the terms of the subtrahend to be changed from + to −, or from − to*

+, and then unite them as in the several cases of addition.

Proof. Subtraction in Algebra is proved just as in Arithmetic, by adding the remainder to the subtrahend; the result should be equal to the minuend.

EXAMPLES.

	(1.)	(2.)	(3.)	(4.)	(5.)
From	$9a$	$7a^2$	$5ax$	$8abx$	$10a^2x$
Take	$4a$	$5a^2$	$3ax$	$3abx$	$7a^2x$
Rem.	$5a$	$2a^2$	$2ax$	$5abx$	$3a^2x$

	(6.)	(7.)	(8.)	(9.)	(10.)
From	$7x$	$5ab$	$6ac$	$9ax$	$7a^2x^2$
Take	$-2x$	$2ab$	$2ac$	$5ax$	$-3a^2x^2$
Rem.	$9x$				

	(11.)	(12.)	(13.)	(14.)	(15.)
From	$4ay$	$12b^2c^2$	$4axy$	$7acy$	$12abx$
Take	$7ay$	$5b^2c^2$	$9axy$	$-8acy$	$7abx$
Rem.	$-3ay$				

	(16.)	(17.)
From	$7y-2b+4x+10$	$9x-3y+7\sqrt{a}+9$
Take	$3y+2b+2x-2$	$5x-7y-3\sqrt{a}-4$
Rem.	$4y-4b+2x+12$	

(18.)

From $6a^2-4ab+7y^3-10$

Take $4a^2-7ab-2y^3-5$

19. From $5xy-4a^2+2b+7$ take $3xy+2a^2-2b+3$.
Ans. $2xy-6a^2+4b+4$.

20. From $5x-2a+5a^2-4$ take $3x+3a+2a^2-6$.
Ans. $2x-5a+3a^2+2$

21. From $5a+2b-5c+6$ take $3a-4b+2c-8$.
Ans. $2a+6b-7c+14$.

22. From $7x-3y+4z+5$ take $3x+5y-2z-4$.
Ans. $4x-8y+6z+9$.

23. From $6ab+3ax-5+7a^2$ take $3ab-4ax+7-3a^2$.
Ans. $3ab+7ax-12+10a^2$.

24 From $9a^2-4ab+8-7b^2$ take $6a^2-8ab-4+3b^2$.
Ans. $3a^2+4ab+12-10b^2$.

25. From $7ax+2ay-10b+7$ take $3ax-5ay-5b+3$.
Ans. $4ax+7ay-5b+4$.

26. From $9ay-2a+4b-5$ take $5ay-4a+6b-3$.
Ans. $4ay+2a-2b-2$.

27. From $6b-4a+7c-5$ take $3b+2a+4c+2$.
Ans. $3b-6a+3c-7$.

28. From $8b-2c^2+3xy+4$ take $3b-5c^2+5xy+7$.
Ans. $5b+3c^2-2xy-3$.

29. From $a^2+2ab+b^2$ take $a^2-2ab+b^2$. *Ans.* $4ab$.

30. From $9x-5a-4y$ take $5x-2a-6y+10$.
Ans. $4x-3a+2y-10$.

31. From $5x-2y+4ab$ take $2x-5y+7ab-4b^2$.
Ans. $3x+3y-3ab+4b^2$.

32. From $3a^3-4a^2x+5ax^2$ take $a^3-2ax^2+x^3$.
Ans. $2a^3-4a^2x+7ax^2-x^3$.

33. From $5a-2b+3c$ take $2a+3b-4y$.
Ans. $3a-5b+3c+4y$.

34. From $5(a+b)+7x-4y$ take $2(a+b)-3x+2y$.
Ans. $3(a+b)+10x-6y$.

35. From $7(x+y)+3a-5b$ take $3(x+y)-2a+3b$.
Ans. $4(x+y)+5a-8b$.

36. From $6(a-b)+3a-5x$ take $2(a-b)+5a+7x$.
Ans. $4(a-b)-2a-12x$.

37. From $5(a+x)+6y+7c$ take $2(a+x)-3y+2c$.
Ans. $3(a+x)+9y+5c$.

38. From $5x^2y-10ab+9$ take $3x^2y+4ab+5$.
Ans. $2x^2y-14ab+4$.

39. From $a^3+3a^2b+3ab^2+b^3$ take $a^3-3a^2b+3ab^2-b^3$.

Ans. $6a^2b+2b^3$.

47. There is one very striking difference between subtraction in Arithmetic and subtraction in Algebra. In Arithmetic all quantities are regarded as positive, and taking one quantity from another necessarily implies *diminution;* in other words, the remainder is always *less* than the minuend. But in Algebra the *difference* between two quantities may often be *greater* than either of the quantities themselves; and this peculiarity is known as the *algebraic difference.*

Ex. 1. What is the difference between +16 d. and −10 d.?

Analysis. Changing the sign of the subtrahend to +10 d., and proceeding as in addition, we find the *algebraic difference* to be +26 d., a quantity greater than either the minuend or the subtrahend. This is easily shown to be in conformity with truth. Suppose +16 d. to represent 16 degrees N. latitude, then −10 d. would properly represent 10 degrees S. latitude; and the difference between 16 degrees N. and 10 degrees S. is evidently 26 degrees.

OPERATION.

$$\begin{array}{rr} & +16\text{ d.} \\ & -10\text{ d.} \\ \hline \textit{Ans.} & +26\text{ d.} \end{array}$$

48. To indicate the subtraction of a polynomial or compound quantity, we enclose the quantity in a parenthesis and prefix the sign −. Thus, if $b^2-2bx+x^2$ were to be taken from $4b^2$, we should write it thus: $4b^2-(b^2-2bx+x^2)$. The − sign before the parenthesis signifies that the *entire* quantity is to be *subtracted,* which is done by removing the parenthesis and changing ALL the signs of the subtrahend. We then have

$$4b^2-(b^2-2bx+x^2)=4b^2-b^2+2bx-x^2=3b^2+2bx-x^2\ \textit{Ans.}$$

NOTE.—The student must not forget that the − sign prefixed to the parenthesis affects the *whole* of the quantity enclosed, and therefore when the subtraction is performed the sign of *every* term is to be changed. This is one of the most difficult things to teach a beginner, and the teacher should dwell on it until it is thoroughly grasped by the pupil.

Ex. 2. What is the value of $3a^2+2b^2-(a^2-b^2+2x^2)$?

Analysis. The $-$ sign before the parenthesis here indicates that $(a^2-b^2+2x^2)$ is to be *subtracted* from $3a^2+2b^2$; removing the parenthesis, and changing the signs of *all* the terms within the parenthesis, our example stands thus:

$$3a^2+2b^2-a^2+b^2-2x^2=2a^2+3b^2-2x^2 \text{ Ans.}$$

3. What is the value of $14x+5b-(6x+10b)$?
Ans. $8x-5b$.

4. What is the value of $5x-2y+3a-(2x-6y+7a)$?
Ans. $3x+4y-4a$.

5. Simplify $4b-2c+3y-(7b+4c-5y+3x)$.
Ans. $-3b-6c+8y-3x$.

6. Simplify $3x+5y-(2x-3y-4z)$. *Ans.* $x+8y+4z$.

7. Simplify $5a+7b-[4a+2b-(a+b)]$.

Analysis. In this example it will be observed there is a *double* parenthesis, or a parenthesis within a bracket. In order to avoid confusion between various pairs of brackets, we use brackets of different *shapes;* we might distinguish by using brackets of the same shape, but of different *sizes.* In such cases it is best to begin by *simplifying* the expression within the brackets. The work would be as follows:

$$5a+7b-[4a+2b-(a+b)]=5a+7b-(4a+2b-a-b).$$

We have removed the parenthesis *within* the brackets, and now we proceed to remove the brackets themselves, just as in the previous examples. Our work now stands:

$$5a+7b-4a-2b+a+b=2a+6b \text{ Ans.}$$

8. Simplify $4a+2x-[5a+3x-(2a+x)]$. *Ans.* a.

9. Simplify $6x+4y-[4x+2y-(2x-2y+3)]$.
Ans. $4x+3$.

10. Simplify $6a+5b-[2a+3b-(4a-2b+5)]$.
Ans. $8a+5$.

11. Simplify $6m-4x+7y-[2m+3x+4y-(3m-4x-3y)]$.
Ans. $7m-11x$.

49. When the quantities in the minuend are different from those in the subtrahend, the subtraction is performed by first changing the signs of the subtrahend, and then connecting the terms as in addition.

Ex. 1. From $2ax-3b+4$ subtract $4ay+2x-3c$.

We simply change the signs of the subtrahend, and connect the terms with those of the minuend by their proper signs.

OPERATION.

$$\begin{array}{r} 2ax-3b+4 \\ \underline{4ay+2x-3c} \\ \textit{Ans. } 2ax-4ay-3b-2x+4+3c. \end{array}$$

2. From $5ab+2c-3$ subtract $6ax-5y+5$.
Ans. $5ab-6ax+2c+5y-8$.

3. From $7b-4x+5y$ take $3a-2b+3$.
Ans. $9b-4x+5y-3a-3$.

50. In subtracting quantities that are partly alike, the result may sometimes be simplified by separating the quantities into factors, and then uniting the difference of the unlike coefficients before the quantity.

Ex. 1. From bx^2 take cx^2.

Analysis. The result will be bx^2-cx^2, and this may be put under the form of $(b-c)x^2$, since bx^2 is equal to b times x^2, and cx^2 is equal to c times x^2; therefore, the difference between bx^2 and cx^2 will be x^2 taken as many times as is expressed by the *difference* of the units in b and c. If $b=7$ and $c=4$, then $bx^2=7x^2$ and $cx^2=4x^2$, and ax^2-bx^2 will equal $7x^2-4x^2=(7-4)x^2=3x^2$.

2. From my subtract ny. *Ans.* $my-ny=(m-n)y$.
3. From ay^2 take by^2. *Ans.* $(a-b)y^2$.
4. From acx take bcx. *Ans.* $(a-b)cx$.
5. From nx take $5x$. *Ans.* $(n-5)x$.
6. From mx take x. *Ans.* $(m-1)x$.
7. From ay take by. *Ans.* $(a-b)y$.
8. From ny take y. *Ans.* $(n-1)y$.
9. From $az+cz$ take nz. *Ans.* $(a+c-n)z$.
10. From $mx+nx$ take px. *Ans.* $(m+n-p)x$.

SECTION IV.

MULTIPLICATION.

51. If 1 yard of calico cost 9 cents, what would be the cost of 5 yards?

Analysis. To answer this question we must *repeat* the 9 cents, or *add* the cost to itself as many times as there are yards; thus,
9 cents + 9 cents + 9 cents + 9 cents + 9 cents = 5 *times* 9 cents = 5 × 9 cents
= 45 cents.
Suppose the calico had cost b cents a yard, then our operation would have been as follows:
b cents + b cents + b cents + b cents + b cents = 5 *times* b cents
= $5b$ cents.

This *repeated addition* of a quantity to itself is called MULTIPLICATION.

52. *Three* terms are used in multiplication; these are, the MULTIPLICAND, the MULTIPLIER and the PRODUCT.

The MULTIPLICAND is the quantity to be repeated.

The MULTIPLIER is the quantity which shows *how* many *times* the multiplicand is to be repeated.

The PRODUCT is the result obtained by the multiplication.

The multiplier and multiplicand are also called *factors.*

53. The SIGN of multiplication is an inclined cross, ×.

54. The *product* must always be like the *multiplicand; i. e.* if we multiply *dollars,* the product must be *dollars;* if we multiply *yards,* the product must be *yards.* Adding a quantity to itself does not change the character of the quantity; and as multiplication is only a short way of performing several additions, the resulting product must always be like the multiplicand.

55. When we multiply one quantity by another, we *repeat* the multiplicand a certain number of times. Hence, the multiplier must *always* be regarded as an abstract number, since it shows how many times the multiplicand is to be repeated.

56. In multiplying two factors it makes no difference which of the two we regard as the multiplicand and which as the multiplier, for in either case the resulting product is the same: $4\times 5=5\times 4$; for in each instance the result is 20 $\cdot$ $a\times b=b\times a=ab$.

In writing the result of a multiplication it is usual to place the numerical coefficient first, and let the letters follow in alphabetic order.

57. Multiplication in Algebra may be divided into three cases.

CASE I.

When the Multiplicand and Multiplier are Simple Quantities.

Ex. 1. If 1 pound of butter cost x cents, what would be the cost of n pounds?

Analysis. If 1 pound cost x cents, n pounds would cost n times as much $=nx$ cents.

OPERATION.

x Multiplicand.
$\underline{n}$ Multiplier.
nx Product.

If $x=25$ and $n=9$, what would be the value of nx?

Ans. ——.

2. What is the product of $4x$ multiplied by n?

Analysis. Since the product of x by n is nx, then the product of $4x$ by n must be *four* times as much $=4nx$. *Ans.* $4nx$.

OPERATION.

$4x$ Multiplicand.
$\underline{n}$ Multiplier.
$4nx$ Product.

3. What is the product of $4x$ multiplied by $7n$?

Analysis. Since the product of $4x$ by n is $4nx$, the product of $4x$ by $7n$ must be *seven* times as much $=28nx$. *Ans.* $28nx$.

OPERATION.

$4x$ Multiplicand.
$\underline{7n}$ Multiplier.
$28nx$ Product.

4. Multiply $7a$ by $9b$. *Ans.* $63ab$.
If $a=2$ and $b=3$, what will be the value of $63ab$?
Ans. ——.

5. Multiply $10c$ by $8m$. *Ans.* $80cm$.
If $c=5$ and $m=4$, what will be the value of $80cm$?
Ans. ——.

6. Multiply $4ab$ by $3xy$. *Ans.* $12abxy$.

7. Multiply $9am$ by $6cx$. *Ans.* ——.

8. Multiply $12abm$ by $7cdy$. *Ans.* $84abcdmy$.

9. Multiply $8bcx$ by $9mpy$. *Ans.* ——.

10. Multiply $7acy$ by $6bmx$. *Ans.* $42abcmxy$.

11. Multiply $13mxy$ by $5abn$. *Ans.* ——.

12. What is the product of $5ab \times 2cm \times 7xy$?
Ans. $70abcmxy$.

13. What is the product of $4cd \times 3mx \times 5ny$? *Ans.* ——.

14. What is the product of $7ax \times 3by \times 6mn$?
Ans. $126abmnxy$.

15. What is the product of $5am \times 3cy \times 6bz$? *Ans.* ——.

16. Multiply y^2 by y^4.

Analysis. y^2 indicates that y is taken as a factor *twice;* and y^4 denotes that y is taken as a factor *four* times; therefore, the product of y^2 by y^4 must contain y as a factor as many times as it is repeated in y^2 and y^4; *i. e.* it must be repeated as a factor *six* times $=y^6$. *Ans.* y^6.

OPERATION.

$$\begin{array}{ll} y^2=yy & \text{Multiplicand.} \\ \underline{y^4=yyyy} & \text{Multiplier.} \\ y^6=yyyyyy & \text{Product.} \end{array}$$

Hence, to multiply powers of the same letter, we have the

RULE.—*Add the exponents.*

17. Multiply $3x^2$ by x^3.

Analysis. According to the preceding example, $x^2 \times x^3 = x^5$; and $3x^2 \times x^3$ must give a product *three* times as great $=3x^5$.

OPERATION.

$$\begin{array}{r} 3x^2=3xx \\ \underline{x^3=\ \ xxx} \\ 3x^5=3xxxxx \end{array}$$

18. Multiply $3m^3$ by $5m^2$.

Analysis. According to the foregoing example, $3m^3 \times m^2 = 3m^5$; and hence, $3m^3 \times 5m^2$ must give a product *five* times as great as before $= 15m^5$.

OPERATION.

$$\begin{array}{l} 3m^3 = 3mmm \\ 5m^2 = 5mm \\ \hline 3m^3 \times 5m^2 = 15mmmmm = 15m^5 \end{array}$$

19. Multiply $5a^2$ by $3a^3$. *Ans.* $15a^5$.

20. Multiply $6b^3$ by $5b^4$. *Ans.* $30b^7$.

21. Multiply $4c^2$ by $3c^3$. *Ans.* ——.

22. Multiply $7a^2b^3$ by $4a^3b^2$. *Ans.* $28a^5b^5$.

23. Multiply $5x^2y^2$ by $3x^3y^2$. *Ans.* ——.

24. Multiply $6ax^4$ by $7a^3x$. *Ans.* $42a^4x^5$.

25. Multiply $4by^2$ by $8b^2y$. *Ans.* ——.

26. Multiply $9a^3bx$ by $5a^2b^2x^2$. *Ans.* $45a^5b^3x^3$.

27. Multiply $5bc^2x^3$ by $6b^2cx^2$. *Ans.* ——.

28. Multiply $12a^2b^3m$ by $7a^2bm^3$. *Ans.* $84a^4b^4m^4$.

29. Multiply $10a^2b^3x$ by $8a^3b^2$. *Ans.* ——.

30. Multiply $12b^2cy^2$ by $6b^2y^3$. *Ans.* $72b^4cy^5$.

31. Multiply $7a^2b^3y$ by $5a^2b^2$. *Ans.* ——.

32. Multiply $5a^m$ by $7a^2$. *Ans.* $35a^{m+2}$.

In this example, as we cannot actually add the exponents of a, we simply *indicate* the addition by connecting the exponents with their proper signs.

33. Multiply $4c^2y^m$ by $9cy^3$. *Ans.* $36c^3y^{m+3}$.

34. Multiply $5b^mc^2$ by $8b^2c^3$. *Ans.* ——.

35. Multiply $7a^nby^2$ by $6ab^2y$. *Ans.* $42a^{n+1}b^3y^3$.

36. Multiply $(a+b)^2$ by $(a+b)^3$. *Ans.* $(a+b)^5$.

In examples like this the multiplication of similar compound quantities is performed just as with *simple* quantities, by *adding* the exponents.

37. Multiply $(a-x)^3$ by $(a-x)^4$. *Ans.* $(a-x)^7$.

38. Multiply $(c+x)^2$ by $(c+x)$. *Ans.* $(c+x)^3$.

39. Multiply $5(a-b)^4$ by $3(a-b)^2$. *Ans.* $15(a-b)^6$.
40. Multiply $a(x-y)^2$ by $b(x-y)$. *Ans.* $ab(x-y)^3$.
41. Multiply $(a+y)^m$ by $(a+y)^m$. *Ans.* $(a+y)^{2m}$.
42. Multiply $(c+d)^{2n}$ by $(c+d)^n$. *Ans.* $(c+d)^{3n}$.
43. Multiply $(a+x)^n$ by $(a+x)$. *Ans.* $(a+x)^{n+1}$.
44. Multiply $5(a+x)^n$ by $4(a+x)^n$. *Ans.* $20(a+x)^{2n}$.
45. Multiply $7(c-x)^4$ by $5(c-x)^3$. *Ans.* $35(c-x)^7$.
46. Multiply $9(a+y)^5$ by $6(a+y)^4$. *Ans.* $54(a+y)^9$.

CASE II.

58. When the Multiplicand is a Compound Quantity, and the Multiplier a Simple Quantity.

Ex. 1. Multiply $a+b+c$ by 5.

OPERATION.

$a+b+c$	Multiplicand.
5	Multiplier.
$5a+5b+5c$	Product.

Analysis. Here the *whole* quantity represented by $(a+b+c)$ is to be *repeated* as many times as there are units in 5 (*i. e. five* times); and this is done by repeating each of the *parts* of which the multiplicand is composed 5 times, giving as the product $5a+5b+5c$.

Ans. $5a+5b+5c$.

2. Multiply $c+d+e$ by m.

OPERATION.

$c+d+e$	Multiplicand.
m	Multiplier.
$cm+dm+em$	Product.

Analysis. Here the *whole* of the multiplicand is to be *repeated* as many times as there are units in m; *i. e.* the units in c must be repeated m times, making cm; the units in d must be repeated m times, making dm; the units in e must be repeated m times, making em; and, finally, the *sum* of these *partial* products will give the *entire* product required. *Ans.* $cm+dm+em$.

From these illustrations we derive the following

RULE.—*Multiply each term of the multiplicand by the multiplier, and add the separate products.*

NOTE.—In Arithmetic we begin at the *right*, and multiply from *right* to *left;* in Algebra it is more convenient to multiply from *left* to *right.*

EXAMPLES.

1. Multiply $x^2+3a^2+4ab+3$ by 4.
Ans. $4x^2+12a^2+16ab+12$.

2. Multiply $3a+5b+6x+7$ by 6. *Ans.* ——

3. Multiply $5y+3ab+2c+4a$ by $3a$.
Ans. $15ay+9a^2b+6ac+12a^2$.

4. Multiply $2ab+5x+5b+7ac$ by $5b$. *Ans.* ——.

5. Multiply $4a^2+7y+9x+3ax$ by $7x$.
Ans. $28a^2x+49xy+63x^2+21ax^2$.

6. Multiply $5c^2+8ab+6y+8ax$ by $9c$. *Ans.* ——.

7. Multiply $a^2+2ab+b^2+c^2$ by a^2.
Ans. $a^4+2a^3b+a^2b^2+a^2c^2$.

8. Multiply $x^2+3xy+2y^2+a^2$ by $2x^2$. *Ans.* ——.

9. Multiply $y^2+4y+3xy+b^2$ by $3y^2$.
Ans. $3y^4+12y^3+9xy^3+3b^2y^2$.

10. Multiply $c^2+2bc+b^2+y^2$ by $5c^2$. *Ans.* ——.

CASE III.

59. When the Multiplicand and Multiplier both are Compound Quantities.

Ex. 1. Multiply $a+b$ by $c+x$.

OPERATION.

$$\begin{array}{l} a+b \\ \underline{c+x\qquad\qquad} \\ ac+bc+ax+bx \end{array}$$

Analysis. The *whole* multiplicand $(a+b)$ is to be *repeated* or *taken* as many times as there are units in $c+x$. Now, $(a+b)$ repeated c times is $ac+bc$ (58), and $(a+b)$ repeated x times is $ax+bx$ (58); adding these *partial* products, we have $ac+bc+ax+bx$ as the *entire* product required.

2. Multiply $m+n$ by $m+n$.

OPERATION.

$$\begin{array}{rrr} m & + & n \\ \underline{m} & \underline{+} & \underline{n} \\ m^2+ & mn & \\ & \underline{+\ mn} & \underline{+n^2} \\ \textit{Ans. } m^2 & +2mn & +n^2 \end{array}$$

Analysis. Here $m+n$ is to be taken as many times as there are units in $m+n$. Now, $(m+n)$ taken m times is m^2+mn (58), and $m+n$ taken n times is $mn+n^2$; the *sum* of these *partial* products will be $m^2+mn+mn+n^2=m^2+2mn+n^2$ by uniting the *like* terms $mn+mn$ into one term.

The algebraic operation here given is the same in principle as that employed in Arithmetic. For example, let it be required to multiply 44 by 33:

Arith. operation.		Alg. operation.
44	=	$40+4$
33	=	$30+3$
132		$1200+120$
132		$+120+12$
1452	=	$1200+240+12=1452$

From the foregoing analyses we derive the following

RULE.—*Multiply each term of the multiplicand by each term of the multiplier, and add the separate products.*

EXAMPLES.

1. Multiply $a+b$ by $c+d$. *Ans.* $ac+bc+ad+bd$.
2. Multiply $a+b$ by $a+b$. *Ans.* $a^2+2ab+b^2$.
3. Multiply $x+y$ by $x+y$. *Ans.* $x^2+2xy+y^2$.
4. Multiply $b+c$ by $b+c$. *Ans.* $b^2+2bc+c^2$.
5. Multiply $2x+3y$ by $3x+2y$. *Ans.* $6x^2+13xy+6y^2$.
6. Multiply $3a^2+5x$ by $2a^2+3x$. *Ans.* $6a^4+19a^2x+15x^2$.
7. Multiply $2ax+4b$ by $3ax+2b$. *A.* $6a^2x^2+16abx+8b^2$.
8. Multiply $3ab+5c$ by $4ab+3c$.
   Ans. $12a^2b^2+29abc+15c^2$.
9. Multiply x^2+xy by x^2+xy. *Ans.* $x^4+2x^3y+x^2y^2$.
10. Multiply $a^2+2ab+b^2$ by $a+b$.
    Ans. $a^3+3a^2b+3ab^2+b^3$.
11. Multiply $b^2+2bx+x^2$ by $b+x$. *A.* $b^3+3b^2x+3bx^2+x^3$.
12. Multiply x^3+x^2+x by x^2+x. *Ans.* $x^5+2x^4+2x^3+x^2$.
13. Multiply y^2+y+1 by $y+1$. *Ans.* y^3+2y^2+2y+1.
14. Multiply $2a^2+3ab+b^2$ by $a+b$.
    Ans. $2a^3+5a^2b+4ab^2+b^3$

SIGNS OF THE PRODUCT.

60. Thus far, our examples in multiplication have been confined exclusively to *positive* quantities. But as Algebra employs both *positive* and *negative* quantities, it becomes necessary to consider the effect on the product when *any* or *all* of the factors are *negative.* A long experience in teaching has convinced the author that the beginner in Algebra should not be required to wrestle with too many difficulties at once. Hence, he has designedly reserved the discussion of the effect of the positive and negative signs upon the product until the student should have become somewhat familiar with the multiplication of positive quantities. We now proceed to this exceedingly important point.

61. According to the definition given in **52**, the *multiplier shows how many times the multiplicand is to be repeated, or added to itself.* It naturally follows that any multiplier with the + sign prefixed indicates that the several repetitions of the multiplicand are to be *added;* but preceded by the − sign, it shows that the repetitions of the multiplicand are to be *subtracted.*

Four different cases may occur, each of which will be considered in turn.

I. What is the product of $+a$ multiplied by $+5$?

OPERATION.

$$\begin{array}{r} +a \\ \underline{+5} \\ +5a \end{array}$$

Analysis. By the conditions of the question we are required to *repeat* $+a$ as many times as there are units in 5; *i. e.* 5 *times.* But as repeating a quantity any number of times does not alter the character of the quantity, it follows that $+a$ repeated $+5$ times will be $+5a$, or $+a\times+5=+5a$. This we express briefly by saying: + multiplied by + produces +.

II. Multiply $-a$ by $+5$.

OPERATION

$$\begin{array}{r} -a \\ \underline{+5} \\ -5a \end{array}$$

Analysis. We are required to repeat $-a$, $+5$ times. Now, it is obvious that a *negative* quantity repeated ever so many times must still be a *negative* quantity (a *debt* repeated however often will still be a *debt*); hence,

$-a$ repeated $+5$ times will be $-5a$, or $-a \times +5 = -5a$. Hence, $-$ multiplied by $+$ produces $-$.

III. Multiply $+a$ by -5.

Analysis. The *repetition* of $+a$, $+5$ times will be $+5a$; but the $-$ sign prefixed to the multiplier shows that the repeated multiplicand is to be *subtracted;* and to *subtract* a quantity is to *change its sign.* Hence, $+5a$ must be changed to $-5a$; *i. e.* $+a \times -5 = -5a$. Hence, $+$ multiplied by $-$ produces $-$.

OPERATION.

$$\begin{array}{r} +a \\ \underline{-5} \\ -5a \end{array}$$

IV. Multiply $-a$ by -5.

Analysis. The repetition of $-a$, 5 times will be $-5a$; but the $-$ sign prefixed to the multiplier, -5, shows that the repeated multiplicand is to be *subtracted,* which is done by *changing the sign* from $-5a$ to $+5a$; *i. e.* $-a \times -5 = +5a$. Hence, $-$ multiplied by $-$ produces $+$.

OPERATION.

$$\begin{array}{r} -a \\ \underline{-5} \\ +5a \end{array}$$

62. These four principles may be briefly expressed thus: In multiplication, LIKE *signs give* $+$, and UNLIKE *signs give* $-$.

63. Every case in multiplication is embraced in the following

RULE.—*Multiply each term of the multiplicand by each term of the multiplier, and connect the partial products by their proper signs.*

NOTE I.—When some of the terms are *alike* they may be united by the rules of addition.

NOTE II.—Equal terms with *unlike* signs balance each other in the product, and disappear.

EXAMPLES.

(1.)

$$\begin{array}{rrrr}
 & 2x^2 & & \\
\end{array}$$

$$\begin{array}{lrrr}
 & 2x & -\ 3y & \\
 & 4x & +\ 5y & \\
\hline
 & 8x^2 & -12xy & \\
 & & +10xy & -15y^2 \\
\hline
\textit{Ans.} & 8x^2 & -\ 2xy & -15y^2
\end{array}$$

(2.)

$$\begin{array}{lrrr}
 & 3ab & -\ 4c & \\
 & 2ab & +\ 7c & \\
\hline
 & 6a^2b^2 & -\ 8abc & \\
 & & +21abc & -28c^2 \\
\hline
\textit{Ans.} & 6a^2b^2 & +13abc & -28c^2
\end{array}$$

(3.)

$$\begin{array}{rl} & y^3+y^2+y \\ & \underline{y^2-y} \\ & y^5+y^4+y^3 \\ & \underline{\quad\; -y^4-y^3-y^2} \\ \textit{Ans.} & y^5 \qquad\qquad -y^2 \end{array}$$

(4.)

$$\begin{array}{rl} & a^2-2ab+b^2 \\ & \underline{a\;-b} \\ & a^3-2a^2b+\;ab^2 \\ & \underline{\quad\; -\;a^2b+2ab^2-b^3} \\ \textit{Ans.} & a^3-3a^2b+3ab^2-b^3 \end{array}$$

5. Multiply $3a+2x$ by $5a-3x$. *Ans.* $15a^2+ax-6x^2$.

6. Multiply $5ab+2y$ by $3ab-4y$. *Ans.* $15a^2b^2-14aby-8y^2$.

7. Multiply $4a^2+3b^2$ by $4a^2-3b^2$. *Ans.* $16a^4-9b^4$

8. Multiply $2x^3+4x^2+8x+16$ by $3x-6$. *Ans.* $6x^4-96$.

9. Multiply $2y^2+2y+5$ by $3y^2-3y$. *Ans.* $6y^4+9y^2-15y$.

10. Multiply $2x^2-3x+2$ by $x-8$. *Ans.* $2x^3-19x^2+26x-16$.

11. Multiply y^2+y+1 by $y-1$. *Ans.* y^3-1.

12. Multiply $1-a+a^2-a^3$ by $1+a$. *Ans.* $1-a^4$.

13. Multiply $b^3+3b^2+9b+27$ by $b-3$. *Ans.* b^4-81.

14. Multiply $a^2+2ab+b^2$ by $a^2-2ab+b^2$. *Ans.* $a^4-2a^2b^2+b^4$.

15. Multiply $2a^3+6a^2+18a+54$ by $2a-6$. *Ans.* $4a^4-324$.

16. Multiply b^2-c^2 by b^2+c^2. *Ans.* b^4-c^4.

17. Multiply $b^2+3bx+2x^2$ by $2b-3x$. *Ans.* $2b^3+3b^2x-5bx^2-6x^3$.

18. Multiply x^2-xy+y^2 by $x+y$. *Ans.* x^3+y^3.

64. In many algebraic calculations it is frequently convenient to *indicate* the multiplication of compound quantities without actually performing the work. This is done by simply enclosing the quantities in a parenthesis, and writing them one after the other. Thus $(a+b)(m+n)$ indicates that $(a+b)$ is to be multiplied by $(m+n)$. When the work indicated is performed by actually multiplying the terms, the expression is said to be *expanded.* Thus, $(a-b)(a-b)$ becomes, when *expanded,* $a^2-2ab+b^2$; $(c+d)(c-d)$ becomes, by expansion, c^2-d^2.

EXAMPLES.

1. Expand $(a+b)(a+b)$. *Ans.* $a^2+2ab+b^2$.
2. Expand $(2x+7)(2x-7)$. *Ans.* $4x^2-49$.
3. Expand $(3y+4)(2y-5)$. *Ans.* $6y^2-7y-20$.
4. Expand $(x^2+x+1)(x-1)$. *Ans.* x^3-1.
5. Expand $(a-x)(b+x)$. *Ans.* $ab+ax-bx-x^2$.
6. Expand $(a+b-c)(a-b+c)$. *Ans.* $a^2-b^2+2bc-c^2$.
7. Expand $(m^2+2mn+n^2)(m-n)$. *Ans.* $m^3+m^2n-mn^2-n^3$.
8. Expand $(a^m+b^m)(a+b)$. *Ans.* $a^{m+1}+ab^m+a^mb+b^{m+1}$.
9. Expand $(a-x)^2$. *Ans.* $a^2-2ax+x^2$.

65. In **61** it was shown that the product of TWO *negative* factors will give a *positive* product; thus, $-a\times -a=+a^2$.

Then, $-a\times -a\times -a=-a^3=$ prod. of *three* neg. fac.
$-a\times -a\times -a\times -a=+a^4=$ " *four* "
$-a\times -a\times -a\times -a\times -a=-a^5=$ " *five* "
$-a\times -a\times -a\times -a\times -a\times -a=+a^6=$ " *six* "

A little inspection will show that the product of an EVEN number of *negative* factors will always be *positive,* but the product of an ODD number of such factors will always be *negative.*

PROMISCUOUS EXAMPLES.

1. Multiply $2x-3y+5ab-4$ by 3. *Ans.* ——.
2. Multiply $3ax-2a^2+7by-5$ by $-4a$. *Ans.* $-12a^2x+8a^3-28aby+20a$.
3. Multiply $5x^2-4ax+6y^2-3$ by $6x$. *Ans.* ——.
4. Multiply $4b+2y-3a+5$ by $-3y$. *Ans.* ——.

5. Multiply $3ab-5ax+2y-3$ by $4a$. *Ans.* ——.

6. Multiply $6ax+2ab-3c+5y$ by $-3x$. *Ans.* ——.

7. Multiply $2a^2-4b^2y+3ac-2c$ by $5c$. *Ans.* ——.

8. Multiply $3a+2x$ by $4a-3x$. *Ans.* $12a^2-xa-6x^2$.

9. Multiply $2x+3y$ by $5x-4y$. *Ans.* $10x^2+7xy-12y^2$.

10. Multiply x^2-2x+3 by $x+4$. *Ans.* $x^3+2x^2-5x+12$.

11. Multiply $2a^2+3a-2$ by $a+3$. *Ans.* $2a^3+9a^2+7a-6$.

12. Multiply $x^2-2bx+b^2$ by $x+b$. *Ans.* $x^3-bx^2-b^2x+b^3$.

13. Multiply $a^2-2ab+b^2$ by $a-b$.
Ans. $a^3-3a^2b+3ab^2-b^3$.

14. Multiply $c^2+3bc-b^2$ by $c+b$. *Ans.* $c^3+4bc^2+2b^2c-b^3$.

15. Multiply $2b^2-3bc+c^2$ by $b-c$.
Ans. $2b^3-5b^2c+4bc^2-c^3$.

16. Multiply a^3+a^2+a by a^2-a. *Ans.* a^5-a^2.

17. Multiply b^3-b^2+b by b^2-b. *Ans.* $b^5-2b^4+2b^3-b^2$.

18. Multiply $2a^2-4ab+2b^2$ by $3a+2b$.
Ans. $6a^3-8a^2b-2ab^2+4b^3$.

19. Multiply $3x^2-2xy+2y^2$ by $2x+2y$.
Ans. $6x^3+2x^2y+4y^3$.

20. Multiply $2c^2+5cy+3y^2$ by $2c-3y$.
Ans. $4c^3+4c^2y-9cy^2-9y^3$.

THEOREMS IN MULTIPLICATION.

66. A THEOREM is a statement of a truth which requires to be proved by a course of reasoning. We propose to present a few theorems which, from their great practical utility, should be thoroughly mastered by the student. He should not be satisfied with being able to apply them practically by the aid of a slate or blackboard, but should become so thoroughly familiar with the principle involved as to produce the result *mentally* when required to do so.

A **PROBLEM** is a question proposed for solution; *i. e.* something to be *done.*

THEOREM I.

67. *The square of the* SUM *of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the second.*

Ex. 1. What is the square of $(a+b)$?

Analysis. Here a is the first quantity, and b the second; $a+b$ is their sum, and in multiplying $(a+b)$ by $(a+b)$ we obtain the square of $(a+b)$, which we find to be $a^2+2ab+b^2$. Now, $a^2=$ the square of the first quantity; $2ab$ is twice the product of the first quantity by the second, and $b^2=$ the square of the second; hence, the *truth* of the theorem.

OPERATION.

$$\begin{array}{l} a+b \\ \underline{a+b} \\ a^2+ab \\ \underline{\quad\;\; +ab+b^2} \\ a^2+2ab+b^2 \end{array}$$

2. What is the square of $3x+2y$?

Analysis. Here $3x$ is the first quantity, and $2y$ the second. The square of $3x=9x^2$; twice the product of $3x$ by $2y$ is $12xy$, and the square of $2y$ is $4y^2$. Hence, $(3x+2y)^2=9x^2+12xy+4y^2$.

EXAMPLES.

3. What is the square of $(a+y)$? *Ans.* $a^2+2ay+y^2$.
4. What is the square of $(b+n)$? *Ans.* ——.
5. What is the square of $(2x+y)$? *Ans.* $4x^2+4xy+y^2$.
6. What is the square of $(3a+b)$? *Ans.* ——.
7. What is the square of $(2m+1)$? *Ans.* $4m^2+4m+1$.
8. What is the square of $(3a+x)$? *Ans.* ——.
9. What is the square of $(3m+2n)$? *Ans.* $9m^2+12mn+4n^2$.
10. What is the square of $(4a+3b)$? *Ans.* ——.
11. Square $(4ab+2x)$. *Ans.* $16a^2b^2+16abx+4x^2$.
12. Square $(5a+2c)$. *Ans.* ——.
13. Square $(3a^2+1)$. *Ans.* $9a^4+6a^2+1$.
14. Square $(4x+2)$. *Ans.* ——.

15. Square $(2a+\frac{1}{2})$. *Ans.* $4a^2+2a+\frac{1}{4}$.
16. Square $(3x+\frac{1}{3})$. *Ans.* $9x^2+2x+\frac{1}{9}$.
17. Square $(4y+\frac{1}{2})$. *Ans.* ——.
18. Square (c^m+1). *Ans.* $c^{2m}+2c^m+1$.

Note.—To show the utility of this theorem, suppose the student is required to expand $(a+b)(a+b)(a+b)$. By the ordinary method he would first multiply $(a+b)$ and $(a+b)$ together, and afterward multiply the product so obtained by $(a+b)$. But, in accordance with the theorem just given, he knows that $(a+b)(a+b)=a^2+2ab+b^2$; hence, he may commence his multiplication at this point, and, multiplying $a^2+2ab+b^2$ by $(a+b)$, he will have the expansion required.

THEOREM II.

68. *The square of the* DIFFERENCE *of two quantities is equal to the square of the first, minus twice the product of the first by the second, plus the square of the second.*

Ex. 1. Find the square of $(a-b)$.

Analysis. Here a and b are the two quantities, and $(a-b)$ is their *difference.* Multiplying $(a-b)$ by $(a-b)$, we obtain $a^2-2ab+b^2$, in which a^2 = the square of the first quantity; $-2ab$ is twice the product of the first quantity by the second, and b^2 = the square of the second; hence, $(a-b)^2=a^2-2ab+b^2$; and therefore the truth of the theorem is manifest.

OPERATION.

$$\begin{array}{r} a - b \\ \underline{a - b } \\ a^2 - ab \\ \underline{ - ab + b^2} \\ a^2 - 2ab + b^2 \end{array}$$

EXAMPLES.

2. Find the square of $(x-y)$. *Ans.* $x^2-2xy+y^2$.
3. Find the square of $(2a-x)$. *Ans.* ——.
4. Find the square of (a^2-2b). *Ans.* $a^4-4a^2b+4b^2$.
5. Find the square of $(2n-1)$. *Ans.* ——.
6. Find the square of $(1-x)$. *Ans.* $1-2x+x^2$.
7. Find the square of $(3a-2y)$. *Ans.* ——.
8. Find the square of $(4a^2-3)$. *Ans.* $16a^4-24a^2+9$.
9. Find the square of $(2a-2c)$. *Ans.* ——.

10. Find the square of $(3m-2)$. *Ans.* $9m^2-12m+4$.
11. Square $(3x-3)$. *Ans.* ——.
12. Square $(2b^2-3)$. *Ans.* $4b^4-12b^2+9$.
13. Square $(4m-1)$. *Ans.* ——.
14. Square (m^2+3c). *Ans.* $m^4+6cm^2+9c^2$.
15. Square $(2a-bc)$ *Ans.* ——.
16. Square $(ab-mn)$. *Ans.* $a^2b^2-2abmn+m^2n^2$.
17. Square $(b-\frac{1}{2})$ *Ans.* ——.
18. Square $(y-\frac{1}{3})$. *Ans.* $y^2-\frac{2y}{3}+\frac{1}{9}$.
19. Square $(c-\frac{1}{5})$. *Ans.* ——.
20. Square (a^n-1). *Ans.* $a^{2n}-2a^n+1$.

THEOREM III.

69. *The product of the* SUM *and the* DIFFERENCE *of two quantities is equal to the difference of their squares.*

Ex. 1. What is the product of $(a+b)$ multiplied by $(a-b)$?

OPERATION.

$$\begin{array}{rr} a & +b \\ a & -b \\ \hline a^2 & +ab \\ & -ab-b^2 \\ \hline a^2 & \quad -b^2 \end{array}$$

Analysis. Multiplying $(a+b)$ by $(a-b)$ in the usual way, we get a^2-b^2, which is obviously the *difference* between the *squares* of the two quantities.

EXAMPLES.

2. Multiply $(x+y)$ by $(x-y)$. *Ans.* x^2-y^2.
3. Multiply $(m+n)$ by $(m-n)$. *Ans.* ——.
4. Multiply $(2a+3y)$ by $(2a-3y)$. *Ans.* $4a^2-9y^2$.
5. Multiply $(3x+4a)$ by $(3x-4a)$. *Ans.* ——.
6. Multiply $(5x+2y)$ by $(5x-2y)$. *Ans.* $25x^2-4y^2$.
7. Multiply $(4b+7c)$ by $(4b-7c)$. *Ans.* ——.
8. Multiply $(6a+1)$ by $(6a-1)$. *Ans.* $36a^2-1$.

9. Multiply $(3x+5y)$ by $(3x-5y)$. *Ans.* ——.

10. Multiply $(2a^2+3c)$ by $(2a^2-3c)$. *Ans.* $4a^4-9c^2$.

11. Multiply $(3x^2+1)$ by $(3x^2-1)$. *Ans.* ——.

12. Multiply $(3ax+2y)$ by $(3ax-2y)$. *Ans.* $9a^2x^2-4y^2$.

13. Multiply $(4am+2x)$ by $4am-2x)$. *Ans.* ——.

14. Multiply $(5bc+3a^2)$ by $(5bc-3a^2)$. *Ans.* $25b^2c^2-9a^4$.

15. Multiply $(2ax+3y)$ by $(2ax-3y)$. *Ans.* ——.

16. Multiply $(3am+\frac{1}{2})$ by $(3am-\frac{1}{2})$. *Ans.* $9a^2m^2-\frac{1}{4}$.

17. Multiply $(4cx+\frac{2}{3})$ by $(4cx-\frac{2}{3})$. *Ans.* ——.

18. Multiply (a^n+1) by (a^n-1). *Ans.* $a^{2n}-1$.

SECTION V.

DIVISION.

70. DIVISION *is the process of finding how many times one quantity is contained in another.*

The DIVIDEND is the quantity to be divided.

The DIVISOR is the quantity by which to divide.

The QUOTIENT is the quantity which shows *how many times* the divisor is contained in the dividend.

The REMAINDER is that which is *sometimes* left after dividing. The remainder, being a part of the dividend, is always of the same kind or denomination as the dividend. Thus, if the dividend is *dollars*, the remainder will be *dollars;* if the dividend is *pounds*, the remainder will be *pounds.* When there is no remainder the division is said to be *exact.*

71. The SIGN of DIVISION is a short horizontal line, with a dot above and another below, ÷; thus, $2a \div 3b$ denotes that $2a$ is to be divided by $3b$.

Division is more frequently indicated by a line, with the dividend above it and the divisor below it; $\frac{3x}{5a}$ denotes that $3x$ is to be divided by $5a$; $3x$ is the dividend, and $5a$ the divisor.

72. Division is the *reverse* of multiplication. In multiplication we usually have *two* factors given, to find their *product*. In division we have the *product* and *one* of the factors, to find the other factor. From this principle we deduce another definition of division—viz.:

DIVISION *consists in finding a quotient which, multiplied by a given divisor, will produce a given dividend.*

Hence, to ascertain how many times one quantity is contained in a second, we merely find what quantity multiplied by the first will produce the second.

Ex. 1. 9 is contained in 72 how many times?

Ans. 8 times.

How do we know this? Because 9 times 8 are 72.

2. $7ax$ is contained in $56a^3x$ how many times?

Ans. $8a^2$ times.

How do we know this? Because $7ax \times 8a^2 = 56a^3x$.

3. Divide $84x^2y$ by $12xy$. *Ans.* $7x$.

How do we know this to be correct?

Because $12xy \times 7x = 84x^2y$.

4. What is the quotient of $54a^2xy$ divided by $9ax$?

Ans. $6ay$.

How do we know that $6ay$ is correct? *Ans.* ——.

5. Divide $77a^2bc$ by $11ab$. *Ans.* $7ac$. Why?

6. What is the quotient of $75a^2xy \div 15ay$?

Ans. $5ax$. Why?

7. What is $\frac{98a^3bc}{14ab}$ equal to? *Ans.* $7a^2c$. Why?

8. What is the quotient of $81x^2y^2b \div 9xy$?

Ans. $9bxy$. Why?

73. The preceding examples illustrate the principle that the actual operation of dividing one quantity by another consists in *cancelling*, or dropping out of the dividend, a factor equal to the divisor; thus,

$$48 \div 8 = \frac{48}{8} = \frac{8 \times 6}{8} = 6;$$

$$40a^2x \div 5ax = \frac{40a^2x}{5ax} = \frac{8a \times 5ax}{5ax} = 8a;$$

$$18a^3 \div 6a = \frac{18a^3}{6a} = \frac{6a \times 3a^2}{6a} = 3a^2.$$

In each of these examples we simply threw out of the dividend a factor equal to the divisor; and this principle holds good *universally*.

EXAMPLES.

1. Divide $15axy$ by $3ay$.	*Ans.* $5x$.
2. Divide $45abcd$ by $9ac$.	*Ans.* $5bd$.
3. Divide $56axbz$ by $8ab$.	*Ans.* ——.
4. Divide $96bcx$ by $12bx$.	*Ans.* $8c$.
5. Divide $40abcf$ by $8bf$.	*Ans.* ——.
6. Divide $63mnp$ by $7mn$.	*Ans.* $9p$.
7. Divide $54aprx$ by $6ap$.	*Ans.* ——.
8. Divide $108bcxz$ by $12bx$.	*Ans.* $9cz$.
9. $132amxy$ by $11ay$.	*Ans.* ——.
10. Divide $7ax$ by $9by$.	*Ans.* $\frac{7ax}{9by}$.

SIGNS IN DIVISION.

74. The rules for the signs in division may be deduced from the principles of multiplication, since the product of the divisor and quotient must always be equal to the dividend.

$$\left.\begin{array}{l} +a \times +b = +ab \\ -a \times +b = -ab \\ +a \times -b = -ab \\ -a \times -b = +ab \end{array}\right\}, \text{ therefore } \left\{\begin{array}{l} +ab \div +b = +a. \\ -ab \div +b = -a. \\ -ab \div -b = +a. \\ +ab \div -b = -a. \end{array}\right.$$

A little examination of the foregoing schedule will show us that the rules for the signs in division are precisely the same as the rules for the signs in multiplication; *i. e. if the divisor and the dividend are both positive, or both negative, the quotient will be positive; but if one is positive, and the other negative, then the quotient will be negative;* or, to express it more concisely, in division LIKE *signs give* +, *and* UNLIKE *signs give* −.

DIVISION OF POWERS.

Ex. 1. Divide b^6 by b^4.

Analysis. We first represent the quotient under the form $\frac{b^6}{b^4}$; but $b^6 = bbbbbb$ (23) and $b^4 = bbbb$; then, *cancelling* equal factors in the dividend and in the divisor, we have $bb = b^2$ as the quotient.

OPERATION.

$$b^6 \div b^4 = \frac{b^6}{b^4} = \frac{bbbbbb}{bbbb} = bb = b^2$$

2. Divide y^5 by y^2.

OPERATION.

$$y^5 \div y^2 = \frac{y^5}{y^2} = \frac{yyyyy}{yy} = yyy = y^3 \text{ Ans.}$$

3. Divide a^4 by a^2.

OPERATION.

$$a^4 \div a^2 = \frac{a^4}{a^2} = \frac{aaaa}{aa} = aa = a^2 \text{ Ans.}$$

A slight inspection of these examples will show us that we might have obtained the correct quotient in each case by simply *subtracting* the exponent of the divisor from that of the dividend. Thus,

In Ex. 1, $\frac{b^6}{b^4}=b^{6-4}=b^2$;

In Ex. 2, $\frac{y^5}{y^2}=y^{5-2}=y^3$;

In Ex. 3, $\frac{a^4}{a^2}=a^{4-2}=a^2$. Hence—

75. A POWER *is divided by another power of the same letter by subtracting the exponent of the divisor from the exponent of the dividend.*

EXAMPLES.

4. Divide $15a^5$ by $5a^3$. *Ans.* $3a^2$.
5. Divide $28x^4$ by $7x^2$. *Ans.* $4x^2$.
6. Divide $42a^5x^4$ by $6a^2x^2$. *Ans.* $7a^3x^2$.
7. Divide $54x^4y^7$ by $9x^2y^3$. *Ans.* $6x^2y^4$.
8. Divide $66b^3c^4$ by $11b^2c^2$. *Ans.* $6bc^2$.

Three different cases may occur in division, each of which will be considered in its order.

CASE I.

76. When the Dividend and the Divisor are both SIMPLE Quantities.

Combining the principles enunciated in 73 and 75, we deduce the following

RULE.—*Divide the coefficient of the dividend by the coefficient of the divisor, and cancel all literal factors that are common to both dividend and divisor.*

EXAMPLES.

1. Divide $24y^5$ by $4y^2$.	*Ans.* $6y^3$.
2. Divide $35x^4$ by $5x$.	*Ans.* $7x^3$.
3. Divide $42abx$ by $7ax$.	*Ans.* $6b$.
4. Divide $36a^3x^2$ by $6ax$.	*Ans.* $6a^2x$.
5. Divide $45a^2bx^3$ by $9ax$.	*Ans.* $5abx^2$.
6. Divide $54x^2y^2z$ by $6xyz$.	*Ans.* $9xy$.
7. Divide $-56a^2bc$ by $8ab$.	*Ans.* $-7ac$.
8. Divide $63a^3b^2x$ by $-9a^2b$.	*Ans.* $-7abx$.
9. Divide $-49x^2y^2$ by $-7xy$.	*Ans.* $7xy$.
10. Divide $72a^5x^4$ by $12a^3x^4$.	*Ans.* $6a^2$.
11. Divide $84a^3x^2y$ by $7a^2x$.	*Ans.* $12axy$.
12. Divide $90b^2c^2x$ by $9bcx$.	*Ans.* $10bc$.
13. Divide $96m^3y^2$ by $12my$.	*Ans.* $8m^2y$.
14. Divide $105b^2c^2y$ by $15bcy$.	*Ans.* $7bc$.
15. Divide $-108a^2bc^2$ by $12\ abc$.	*Ans.* $-9ac$.
16. Divide $114n^2xy$ by $19ny$.	*Ans.* $6nx$.
17. Divide $121a^3b^4x^2$ by $11a^2b^2x$.	*Ans.* $11ab^2x$.

CASE II.

77. When the Dividend is a Compound Quantity, and the Divisor a Simple Quantity.

If we multiply $3a-4b+5c-2x$ by $3x$, the product will be $9ax-12bx+15cx-6x^2$ (58). Hence, it follows that if $9ax-12bx+15cx-6x^2$ be divided by $3x$, the quotient will be $3a-4b+5c-2x$.

Hence, to divide a compound quantity by a simple quantity we have the

RULE.—*Divide each term of the dividend separately by the divisor, and connect the quotients by their proper signs.*

EXAMPLES.

1. Divide $20ax-8ay+16a^2-24ab+12ac$ by $4a$.

OPERATION.

$$\begin{array}{l} 4a\underline{)20ax-8ay+16a^2-24ab+12ac} \\ \quad\;\; 5x-\;2y+\;4a-6b\;\;+\;3c \text{ Ans.} \end{array}$$

Analysis. We write the divisor at the *left* of the dividend, just as in short division in Arithmetic, and say: $4a$ is contained in $20ax$, $5x$ times; in $-8ay$, $-2y$ times; in $16a^2$, $4a$ times; in $-24ab$, $-6b$ times; in $12ac$, $3c$ times. Hence, the entire quotient is $5x-2y+4a-6b+3c$.

2. Divide $12x^2-6xy+9ax-15bx+9cx$ by $3x$.
Ans. $4x-2y+3a-5b+3c$.

3. Divide $12ay-8by+16cy-20y^2$ by $4y$.
Ans. $3a-2b+4c-5y$.

4. Divide $10abx-15a^2b+25ab^2-20abc$ by $5ab$.
Ans. $2x-3a+5b-4c$.

5. Divide $18ax^2-12axy+18a^2x-5cy$ by $6ax$.
Ans. $3x-2y+3a-\frac{5cy}{6ax}$.

In this last example the divisor not being contained in the last term of the dividend, we *indicate* the division simply by writing the divisor under the dividend in the form of a fraction, and annex it to the preceding part of the quotient.

6. Divide $-28b^2c+35bc^2-21bcx+14bcy$ by $-7bc$.
Ans. $4b-5c+3x-2y$.

7. Divide $27x^2y-18bxy+36xy^2-45xy$ by $9xy$.
Ans. $3x-2b+4y-5$.

8. Divide $16a^2x-24ax+32ax^2-7by$ by $8ax$.
Ans. $2a-3+4x-\frac{7by}{8ax}$.

9. Divide $24x^2y-36xy+60xy^2-72x^3y^2$ by $12xy$.
Ans. $2x-3+5y-6x^2y$.

10. Divide $21a^2b^2-35ab+56a^3b-63ab^3$ by $7ab$.
Ans. $3ab-5+8a^2-9b^2$.

CASE III.

78. When the Dividend and Divisor are both Compound Quantities.

Ex. 1. Divide $a^3+3a^2x+3ax^2+x^3$ by $(a+x)$.

OPERATION.

$$\begin{array}{lll}
\textbf{Divisor.} & \textbf{Dividend.} & \textbf{Quotient.} \\
a+x) & a^3+3a^2x+3ax^2+x^3 & (a^2+2ax+x^2 \\
 & \underline{a^3+\ \ a^2x\qquad\qquad} & \\
 & \qquad\ 2a^2x+3ax^2 & \\
 & \qquad\ \underline{2a^2x+2ax^2\qquad} & \\
 & \qquad\qquad\qquad ax^2+x^3 & \\
 & \qquad\qquad\qquad \underline{ax^2+x^3} &
\end{array}$$

Analysis. We proceed substantially as in long division in Arithmetic, writing the divisor on the *left* of the dividend, and, for convenience, arranging the letters of the divisor and the dividend in the same order, so that the highest power of the same letter in each shall come first, the next highest next, and so on. We then divide the first term of the dividend by the first term of the divisor, and we find that a (the first term of the divisor) is contained in a^3, a^2 times; we place a^2 on the *right*, as the first term of the quotient. We then multiply the *whole* divisor, $(a+x)$, by a^2, and place the product, (a^3+a^2x), under the dividend; then subtracting this product and bringing down another term, we have $2a^2x+3ax^2$ as a new dividend. Dividing the first term of this new dividend by the first term of the divisor as before, we get $2ax$ as the second term of our quotient. Multiplying the *whole* divisor by $2ax$, we obtain the product, $2a^2x+2ax^2$, which we place under the new dividend; then subtracting this product and bringing down another term, we have ax^2+x^3 as a *third dividend.* Dividing as before, we obtain x^2 as the *third* term of our quotient, and multiplying our whole divisor by this term of the quotient, we get ax^2+x^3, and subtracting this product from the last dividend, we find there is no remainder, and the work is complete.

What we have said may be summed up in the following

RULE.—*I. Arrange the divisor and dividend with reference to the powers of some one letter.*

II. Divide the first term of the dividend by the first term of the divisor, and the result will be the first term of the quotient.

III. Multiply the whole divisor by this quotient;

subtract the product from the dividend, and to the remainder bring down as many terms as may be necessary, to form a new dividend.

IV. Divide the first term of the new dividend by the first term of the divisor; the result will be the second term of the quotient.

V. Multiply the divisor by the second term of the quotient, subtract the product, and proceed as before, until all the terms of the dividend have been brought down.

NOTE.—If there be a remainder after all the terms of the dividend have been brought down, it must be placed over the divisor, and annexed to the quotient with its proper sign.

EXAMPLES.

2. Divide $a^2+2ax+x^2$ by $(a+x)$.

OPERATION.

$$\begin{array}{l} a+x)a^2+2ax+x^2(a+x \\ \underline{a^2+\ ax\quad} \\ ax+x^2 \\ \underline{ax+x^2} \end{array}$$

3. Divide $b^2-2bc+c^2$ by $(b-c)$. *Ans.* $b-c$.
4. Divide $4x^2-12xy+9y^2$ by $(2x-3y)$. *Ans.* $2x-3y$.
5. Divide $x^2+8x+15$ by $(x+5)$. *Ans.* $x+3$.
6. Divide $6a^2+19ax+10x^2$ by $(2a+5x)$. *Ans.* $3a+2x$.
7. Divide $12y^2-7y-10$ by $(4y-5)$. *Ans.* $3y+2$.
8. Divide $6x^2+x-12$ by $(3x-4)$. *Ans.* $2x+3$.
9. Divide $10x^2-31x+15$ by $(5x-3)$. *Ans.* $2x-5$.
10. Divide $20y^2+19y+3$ by $(4y+3)$. *Ans.* $5y+1$.
11. Divide $4x^2-49$ by $(2x-7)$. *Ans.* $2x+7$.
12. Divide $x^3+2x^2-5x+12$ by $(x+4)$. *Ans.* x^2-2x+3.
13. Divide x^4-16 by $(x-2)$. *Ans.* x^3+2x^2+4x+8.
14. Divide $9y^3+6y^2-5y+4$ by $(3y+4)$. *Ans.* $3y^2-2y+1$.
15. Divide y^4-256 by $(y-4)$. *Ans.* $y^3+4y^2+16y+64$.

16. Divide $2x^3+11x^2+6x-9$ by $(2x+3)$.

Ans. x^2+4x-3.

17. Divide $3x^3-13x^2-21x+15$ by (x^2-6x+3).

Ans. $3x+5$.

18. Divide $15y^3-y^2+19y+15$ by $(3y^2-2y+5)$.

Ans. $5y+3$.

79. Ex. 1. Divide (a^3-x^3) by $(a-x)$.

OPERATION.

$$\begin{array}{r l}
a-x)\,a^3-x^3 & (a^2+ax+x^2 \\
\underline{a^3-a^2x} & \\
a^2x-x^3 & \\
\underline{a^2x-ax^2} & \\
ax^2-x^3 & \\
\underline{ax^2-x^3} &
\end{array}$$

NOTE.—In dividing, special attention must be paid to the management of the signs.

2. Divide (a^4-x^4) by $(a-x)$. *Ans.* $a^3+a^2x+ax^2+x^3$.

3. Divide (a^5-x^5) by $(a-x)$. *Ans.* $a^4+a^3x+a^2x^2+ax^3+x^4$.

4. Divide (a^6-x^6) by $(a-x)$.

Ans. $a^5+a^4x+a^3x^2+a^2x^3+ax^4+x^5$.

From the last four examples we learn this important truth:

That the difference of the same powers of two quantities is always divisible by the difference of the two quantities.

The student cannot fail to notice the *symmetry* of these quotients.

I. In each instance the *number* of terms in the quotient is the same as the exponent of the powers in the dividend.

II. The *sum* of the exponents in each term of the quotient is always 1 *less* than the exponent of the powers.

II. The exponent of the first quantity in the quotient *decreases* regularly by 1, while that of the second *increases* regularly by 1.

IV. The signs of *all* the terms are *positive*.

By carefully noting the points we have mentioned the student can at once write out any quotient of this character, without the necessity of an actual division. We append a few examples as an exercise for the student:

5. Divide (a^7-x^7) by $(a-x)$. *Ans.* ——.

6. Divide (a^8-x^8) by $(a-x)$. *Ans.* ——.

80. Ex. 1. Divide (a^2-x^2) by $(a+x)$.

OPERATION.

$$\begin{array}{l} a+x)a^2-x^2(a-x \\ \underline{\quad a^2+ax} \\ \qquad -ax-x^2 \\ \qquad \underline{-ax-x^2} \end{array}$$

2. Divide (a^4-x^4) by $(a+x)$. *Ans.* $a^3-a^2x+ax^2-x^3$.

3. Divide (a^6-x^6) by $(a+x)$.
Ans. $a^5-a^4x+a^3x^2-a^2x^3+ax^4-x^5$.

4. Divide (a^8-x^8) by $(a+x)$.
Ans. $a^7-a^6x+a^5x^2-a^4x^3+a^3x^4-a^2x^5+ax^6-x^7$.

From the last four examples we deduce another general principle—viz.:

The difference of any two equal powers of two quantities is divisible by the SUM *of the quantities when the exponent of the power is an* EVEN *number.*

The student will notice the same *symmetrical* character in these last quotients as in those of **79**, with the single exception that the signs are alternately positive and negative.

81. Ex. 1. Divide (a^3+x^3) by $(a+x)$.

OPERATION.

$$\begin{array}{l} a+x)a^3+x^3(a^2-ax+x^2 \\ \underline{\quad a^3+a^2x} \\ \qquad -a^2x+x^3 \\ \qquad \underline{-a^2x-ax^2} \\ \qquad\qquad ax^2+x^3 \\ \qquad\qquad \underline{ax^2+x^3} \end{array}$$

2. Divide (a^5+x^5) by $(a+x)$.
Ans. $a^4-a^3x+a^2x^2-ax^3+x^4$.

3. Divide (a^7+x^7) by $(a+x)$.
Ans. $a^6-a^5x+a^4x^2-a^3x^3+a^2x^4-ax^5+x^6$.

From the foregoing examples we deduce another general principle—viz.:

The SUM *of any two equal powers of different quantities is divisible by the* SUM *of the quantities when the exponent of the power is an* ODD *number.*

The student will notice that the quotients in the last several examples follow exactly the same law as those in **80.**

Let the student write out the quotients to the following examples, without an actual division:

4. Divide (a^9+x^9) by $(a+x)$. *Ans.* ——.

5. Divide $(a^{11}+x^{11})$ by $(a+x)$. *Ans.* ——.

PROMISCUOUS EXAMPLES.

1. Divide $5x^2-15xy+20ax-30cx+5x$ by $5x$.
Ans. ——.

2. $14axy-21a^2x^2+35abx-42a^3+56ax$ by $-7ax$.
Ans. ——.

3. Divide $18bxy-27abxy+45b^2y^2-54b^3y$ by $9by$.
Ans. ——.

4. Divide $b+d+bcn+dcn$ by $(b+d)$. *Ans.* $1+cn$.

5. Divide $1-3y+3y^2-y^3$ by $(1-y)$. *Ans.* $1-2y+y^2$.

6. Divide $2x^3+5x^2-7x+20$ by $(x+4)$. *Ans.* $2x^2-3x+5$.

7. Divide $y^3+3y^2z+3yz^2+z^3$ by $(y+z)$. *Ans.* $y^2+2yz+z^2$.

8. Divide $2b^3-3b^2-14b+15$ by $(b-3)$.
Ans. $2b^2+3b-5$.

9. Divide $10x^3+7x^2-6x+9$ by $(2x+3)$.
Ans. $5x^2-4x+3$.

10. Divide x^3-1 by $(x-1)$. *Ans.* x^2+x+1.

11. Divide $4x^4-64$ by $(2x-4)$. *Ans.* $2x^3+4x^2+8x+16$.

12. Divide a^4-x^4 by $(a-x)$. *Ans.* $a^3+a^2x+ax^2+x^3$.

13. Divide $3x^4-243$ by $(3x-9)$. *Ans.* $x^3+3x^2+9x+27$.

14. Divide $15y^3+19y^2-y+15$ by $(3y+5)$. *Ans.* $5y^2-2y+3$.

15. Divide $9y^3-7y+12$ by $(3y+4)$. *Ans.* $3y^2-4y+3$.

16. Divide $8y^3+64$ by $(4y+8)$. *Ans.* $2y^2-4y+8$.

17. Divide $6x^3+48$ by $(2x+4)$ *Ans.* $3x^2-6x+12$.

18. Divide $27x^3+125$ by $(3x+5)$. *Ans.* $9x^2-15x+25$.

19. Divide x^4-y^4 by $(x-y)$. *Ans.* $x^3+x^2y+xy^2+y^3$.

20. Divide x^5+y^5 by $(x+y)$. *Ans.* $x^4-x^3y+x^2y^2-xy^3+y^4$.

21. Divide m^4-n^4 by $(m+n)$ *Ans.* $m^3-m^2n+mn^2-n^3$.

GENERAL PRINCIPLES OF DIVISION.

82. In division the value of the quotient depends on both the dividend and the divisor. If either one of these remains unaltered, a change in the other produces a change in the quotient.

83. If the divisor remains the same, multiplying the dividend by any quantity is, in effect, multiplying the quotient by the same quantity, thus,

$$20 \div 4 = 5,$$
$$60 \div 4 = 15;$$

i. e. multiplying the dividend by 3 multiplies the quotient by 3

$$\text{Or,}\quad mn \div n = m,$$
$$cmn \div n = cm;$$

i. e. multiplying the dividend by c multiplies the quotient by c.

84. If the divisor remains the same, dividing the dividend by any quantity is dividing the quotient by the same quantity; thus,

$$36 \div 6 = 6,$$
$$12 \div 6 = 2;$$

i. e. dividing the dividend by 3 divides the quotient by 3.

$$\text{Or, } cmn \div m = cn,$$
$$mn \div m = n;$$

i. e. dividing the dividend by c divides the quotient by c.

85. If the dividend remains the same, multiplying the divisor by any quantity is dividing the quotient by the same quantity; thus,

$$18 \div 3 = 6,$$
$$18 \div 9 = 2;$$

i. e. multiplying the divisor by 3 divides the quotient by 3.

$$\text{Or, } cmn \div m = cn,$$
$$cmn \div cm = n;$$

i. e. multiplying the divisor by c divides the quotient by c.

86. If the dividend remains the same, dividing the divisor by any quantity is multiplying the quotient by the same quantity; thus,

$$24 \div 6 = 4,$$
$$24 \div 2 = 12;$$

i. e. dividing the divisor by 3 multiplies the quotient by 3.

$$\text{Or, } cmn \div cm = n,$$
$$cmn \div m = cn;$$

i. e. dividing the divisor by c multiplies the quotient by c.

87. Since multiplying the dividend multiplies the quotient (**83**), and multiplying the divisor divides the quotient (**85**), it follows that multiplying both dividend and divisor by the same quantity does not alter the quotient; thus,

$$12 \div 3 = 4,$$
$$36 \div 9 = 4.$$

$$\text{Or,}\quad mn \div m = n,$$
$$cmn \div cm = n.$$

88. Since dividing the dividend divides the quotient (84), and dividing the divisor multiplies the quotient (86), it follows that if the dividend and divisor are both divided by the same quantity, the quotient is not altered; thus,

$$24 \div 6 = 4,$$
$$8 \div 2 = 4.$$
$$\text{Or,}\quad cmn \div cm = n,$$
$$mn \div m = n.$$

NEGATIVE EXPONENTS.

89. Not only do we have negative coefficients in Algebra, but we also have negative *exponents.* The author has intentionally reserved the discussion of *negative* exponents until the student should have become somewhat familiar with the management of *positive* exponents.

90. Negative exponents originate in an extension of the principle of division of powers.

In 75 it was shown that *powers of the same letter or quantity are divided by subtracting the exponent of the divisor from the exponent of the dividend.*

$$\text{Thus,}\quad a^5 \div a^3 = a^{5-3} = a^2,$$
$$a^5 \div a^4 = a^{5-4} = a^1, \text{ or } a,$$
$$a^5 \div a^5 = a^{5-5} = a^0.$$

But any quantitity divided by itself is equal to 1; *i. e.* $a^5 \div a^5 = \dfrac{a^5}{a^5} = 1$; hence, it follows that $a^0 = 1$.

In the same manner, $x^0 = 1$,

$$y^0 = 1,$$
$$10^0 = 1.$$

Any quantity having the exponent 0 is always equal to 1.

91. Again: $a^5 \div a^6 = \frac{a^5}{a^6} = a^{5-6} = a^{-1}$.

But a fraction is reduced to lower terms by dividing both its numerator and its denominator by the same quantity. Dividing the fraction $\frac{a^5}{a^6}$ by a^5, we have $\frac{1}{a}$. Therefore,

$$a^{-1} = \frac{1}{a}.$$

Again: $a^5 \div a^7 = \frac{a^5}{a^7} = a^{5-7} = a^{-2}$.

But reducing the fraction $\frac{a^5}{a^7}$ to its lowest terms, we have $\frac{1}{a^2}$; therefore, $a^{-2} = \frac{1}{a^2}$.

Upon the same principle, $a^{-3} = \frac{1}{a^3}$, $a^{-4} = \frac{1}{a^4}$, $a^{-n} = \frac{1}{a^n}$.

Any quantity with a negative exponent is equal to unity, or 1 divided by the same quantity with a positive exponent; or, in other words, quantities with negative exponents are the *reciprocals* of the same quantities with positive exponents.

92. Powers with *negative* exponents are multiplied in the same manner as powers with *positive* exponents; *i. e.* by *adding the exponents.*

EXAMPLES.

1. Multiply x^{-3} by x^{-2}. *Ans.* $x^{-5} = \frac{1}{x^5}$.

2. Multiply x^5 by x^{-3}. *Ans.* x^2.

3. Multiply y^{-5} by y^3. *Ans.* $y^{-2} = \frac{1}{y^2}$.

4. Multiply a^{-4} by a^{-3}. *Ans.* $a^{-7} = \frac{1}{a^7}$.

5. Multiply a^n by a^{-2}. *Ans.* a^{n-2}.

6. Multiply x^3 by x^{-6}. *Ans.* $x^{-3}=\frac{1}{x^3}$.

7. Multiply a^2x^{-5} by $a^{-4}x^2$. *Ans.* $a^{-2}x^{-3}=\frac{1}{a^2x^3}$.

8. Multiply a^3 by a^{-3}. *Ans.* $a^0=1$.

93. Division of powers with *negative* exponents is performed exactly as in the case of powers with *positive* exponents; *i. e. by subtracting the exponent of the divisor from the exponent of the dividend.*

EXAMPLES.

1. Divide a^5 by a^{-3}.

Analysis. $a^5\div a^{-3}=a^5\div\frac{1}{a^3}=a^5\times\frac{a^3}{1}=a^8$.

By subtracting the exponent of the divisor from that of the dividend, we obtain the same result. *Ans.* a^8.

2. Divide x^4 by x^{-3}. *Ans.* x^7.

3. Divide a^{-5} by a^{-3}. *Ans.* a^{-2}.

4. Divide y^{-6} by y^{-3}. *Ans.* $y^{-3}=\frac{1}{y^3}$.

5. Divide x^3 by x^{-3}. *Ans.* x^6.

6. Divide a^{-2} by a^4. *Ans.* a^{-6}.

7. Divide y^5 by y^{-2}. *Ans.* y^7.

8. Divide a^3x^{-2} by $a^{-2}x^3$. *Ans.* $a^5x^{-5}=\frac{a^5}{x^5}$.

9. Divide $(a+b)^5$ by $(a+b)^{-2}$. *Ans.* $(a+b)^7$.

10. Divide $(x-y)^{-5}$ by $(x-y)^{-2}$. *Ans.* $(x-y)^{-3}$.

SECTION VI.

RESOLVING INTO FACTORS.

94. To *resolve a quantity into its factors* is to find what quantities multiplied together will produce the given quantity. In algebraic calculations there is frequent occasion for resolving quantities into factors. Compound quantities especially are frequently rendered much more convenient for calculation by resolving them into factors. Several Cases may occur, the more important of which will now be examined in their order.

CASE I.

95. To Resolve Monomials, or Simple Quantities, into Factors.

A little inspection is usually all that is necessary in the case of simple quantities, because the factors are mostly *letters*, and are in a great measure *distinct* from each other. Simple quantities can frequently be resolved into factors in quite a variety of methods.

Ex. 1. Resolve $12ax$ into two factors.

This can be done in several different ways; thus,

$$12ax, = 12a \times x, = 12x \times a, = 6a \times 2x, = 6x \times 2a, = 3a \times 4x, = 3x \times 4a,$$
$$= 12 \times ax, = 3 \times 4ax, = 4 \times 3ax, = 2 \times 6ax, = 6 \times 2ax.$$

Thus, we see that this very simple monomial can be resolved into *two* factors by no less than *eleven* different methods.

2. Resolve $14mx$ into two factors in all possible ways.
Ans. The methods are 7.

3. Resolve $18ay$ into two factors in all possible ways.
Ans. The methods are 9

CASE II.

96. To Resolve a Compound Quantity into Two Factors, One of which shall be a Simple Quantity.

Ex. 1. Resolve $3ax-6bx+9cx$ into two factors.

Analysis. A little inspection shows us that $3x$ is a *common* factor of all the terms, and dividing $(3ax-6bx+9cx)$ by the common factor, $3x$, we obtain $(a-2b+3c)$ as the other factor. Hence, $3ax-6bx+9cx=3x(a-2b+3c)$.

OPERATION.

$$3x\underline{)3ax-6bx+9cx}$$
$$a-2b\ \ +3c$$

2. Resolve $2acx-4a^2c+8acy$ into two factors.

Analysis. By inspection we see that $2ac$ is a *common* factor of all the terms, and dividing the given quantity, $2acx-4a^2c+8acy$, by $2ac$, we obtain $(x-2a+4y)$ as the other factor. Hence, $2acx-4a^2c+8acy=2ac(x-2a+4y)$.

OPERATION.

$$2ac\underline{)2acx-4a^2c+8acy}$$
$$x-2a\ \ +4y$$

Ans. $2ac(x-2a+4y)$.

From these illustrations we deduce the following

RULE.—*Divide by the greatest factor common to all the terms, then enclose the quotient in a parenthesis as one factor, and write the divisor as the other factor.*

3. Resolve $(3a^2-6ax)$ into two factors.

Ans. $3a(a-2x)$.

4. Resolve $(5x^2-10bx)$ into two factors.

Ans. $5x(x-2b)$.

5. Resolve (x^2+x) into two factors. *Ans.* $x(x+1)$.

6. Resolve $(x^2-2ax+x)$ into its factors.

Ans. $x(x-2a+1)$.

7. Resolve $(y^2+2xy-y)$ into its factors. *Ans.* ——.

8. Resolve $(x^3+2ax^2-x^2)$ into its factors.

Ans. $x^2(x+2a-1)$.

9. Resolve (x^2+xy+x) into its factors. *Ans.* ——.

10. Resolve $(x^2y^2+2xy^2+xy)$ into its factors.
Ans. $xy(xy+2y+1)$.

11. Resolve $(3a^2-6ax+3a)$ into its factors. *Ans.* ——.

12. Resolve $(2abx+4a^2x^2-6ax)$ into its factors.
Ans. $2ax(b+2ax-3)$.

13. Resolve $(x^2+3ax-x)$ into its factors. *Ans.* ——.

14. Resolve $(5a^2x-2a^2x^2+3ax)$ into its factors.
Ans. $ax(5a-2ax+3)$.

15. Resolve $(3acy-c^2y^2+5bcy)$ into its factors. *Ans.* ——.

16. Resolve $(4axy-2x^2y^2+4x^2y-6xy)$ into its factors.
Ans. $2xy(2a-xy+2x-3)$.

17. Resolve $(3a^2c-6a^2c^2+9ac^2-3ac)$ into its factors.
Ans. ——.

CASE III.

97. To Resolve a TRINOMIAL into Two Equal BINOMIAL Factors.

Ex. 1. Resolve $a^2+2ab+b^2$ into equal factors.

Analysis. A little inspection will show us that a^2 is the *square* of a; that b^2 is the *square* of b, and that $2ab$ is *twice* the product of a and b; and therefore $(a+b)^2$, which is equal to $(a+b)(a+b)$, will give the trinomial $a^2+2ab+b^2$. This is the *reverse* of Theorem I. (67).

Such trinomials as these can be resolved into equal binomial factors, and for this purpose we employ the following

RULE.—*Take the square roots of the two terms which are complete squares, and form two binomial factors by connecting the two square roots by the sign of the other term.*

2. Resolve $4b^2-8bc+4c^2$ into equal factors.

Analysis. The square root of $4b^2$ is $2b$; the square root of $4c^2$ is $2c$; then forming *two* binomials, by connecting the two square roots by the sign of $-8bc$ (the remaining term), we have $(2b-2c)(2b-2c)$.
Ans. $(2b-2c)(2b-2c)$.

EXAMPLES.

3. Resolve $x^2+2xy+y^2$ into equal factors. *Ans.* $(x+y)(x+y)$.

4. Resolve $a^2+2ax+x^2$ into equal factors. *Ans.* ——.

5. Resolve $x^2-2xy+y^2$ into equal factors. *Ans.* $(x-y)(x-y)$.

6. Resolve $a^2-2ac+c^2$ into factors. *Ans.* ——.

7. Resolve $4a^2+12ay+9y^2$ into factors. *Ans.* $(2a+3y)(2a+3y)$.

8. Resolve $9x^2-12xy+4y^2$ into factors. *Ans.* ——.

9. Resolve $16b^2+40bx+25x^2$ into factors. *Ans.* $(4b+5x)(4b+5x)$.

10. Resolve $25x^2-60xy+36y^2$ into factors. *Ans.* ——.

11. Resolve x^2-2x+1 into factors. *Ans.* $(x-1)(x-1)$.

12. Resolve y^2-4y+4 into factors. *Ans.* ——.

13. Resolve $4x^2y^2-12axy+9a^2$ into factors. *Ans.* $(2xy-3a)(2xy-3a)$.

14. Resolve $16a^2c^2+40acy+25y^2$ into factors. *Ans.* ——.

15. Resolve $x^2+x+\frac{1}{4}$ into factors. *Ans.* $(x+\frac{1}{2})(x+\frac{1}{2})$.

16. Resolve $y^2-y+\frac{1}{4}$ into factors. *Ans.* $(y-\frac{1}{2})(y-\frac{1}{2})$.

17. Resolve $a^2+\dfrac{2a}{3}+\frac{1}{9}$ into factors. *Ans.* $(a+\frac{1}{3})(a+\frac{1}{3})$.

18. Resolve $m^2-\dfrac{2m}{3}+\frac{1}{9}$ into factors. *Ans.* ——.

98. Trinomials frequently occur which we are required to resolve into *three* factors, two of which are binomials, and the third a monomial. This operation may be readily effected by combining the principles of Case II. and Case III. By *inspection*, we first find the monomial factor, and divide the trinomial by this factor; then we proceed with the quotient exactly as in Case III.

Ex. 1. Resolve $bx^2+2bxy+by^2$ into three factors.

Analysis. By inspection we see that b is a common factor of all the terms; and, dividing $bx^2+2bxy+by^2$ by b, we obtain $(x^2+2xy+y^2)$ as the other factor. Our trinomial can now be placed under the form of the two factors, $b(x^2+2xy+y^2)$. Then proceeding as in Case III., we resolve $(x^2+2xy+y^2)$ into the two factors, $(x+y)(x+y)$; and now our whole trinomial is resolved into the three factors, $b(x+y)(x+y)$.

EXAMPLES.

2. Resolve $(a^2x+2acx+c^2x)$ into three factors.
Ans. $x(a+c)(a+c)$.

3. Resolve $(x^2y-2xy+y)$ into three factors.
Ans. $y(x-1)(x-1)$.

4. Resolve $(2a^2y+4amy+2m^2y)$ into three factors.
Ans. $2y(a+m)(a+m)$.

5. Resolve $(4a^2x-12ax^2+9x^3)$ into three factors.
Ans. $x(2a-3x)(2a-3x)$.

6. Resolve $(4y^3+4y^2+y)$ into three factors.
Ans. $y(2y+1)(2y+1)$.

7. Resolve $(45a^2n+60anx+20nx^2)$ into three factors.
Ans. $5n(3a+2x)(3a+2x)$.

CASE IV.

99. To Resolve a BINOMIAL into Two Binomial Factors.

Ex. 1. Resolve a^2-b^2 into factors.

Analysis. By recurring to **69**, we there find that $(a+b)(a-b)=a^2-b^2$. Hence, it is obvious that the factors of (a^2-b^2) are $(a+b)$ and $(a-b)$. But a is the *square root* of a^2, and b is the *square root* of b^2.

Hence, when a binomial represents the *difference* between the *squares* of two quantities, we can resolve it into two factors by the following

RULE.—*Take the square roots of the two squares, and connect one pair of the roots by the + sign, and the other pair by the − sign.*

EXAMPLES.

2. Resolve $(16a^2-25y^2)$ into factors.

Analysis. The square root of $16a^2=4a$; the square root of $25y^2=5y$; connecting one pair of the roots by +, and the other pair by −, we have $(4a+5y)(4a-5y)$. *Ans.* $(4a+5y)(4a-5y)$.

3. What are the factors of (x^2-a^2)?
Ans. $(x+a)(x-a)$.

4. Resolve (a^2-y^2) into factors. *Ans.* ——.

5. What are the factors of (m^2-n^2)?
Ans. $(m+n)(m-n)$.

6. Resolve (b^2-c^2) into factors. *Ans.* ——.

7. What are the factors of $(4x^2-9a^2)$?
Ans. $(2x+3a)(2x-3a)$.

8. Resolve $(9b^2-16y^2)$ into factors. *Ans.* ——.

9. What are the factors of $(25a^2-1)$?
Ans. $(5a+1)(5a-1)$.

10. Resolve $4x^2-1$ into factors. *Ans.* ——.

11. What are the factors of $(36y^2-\frac{1}{4})$?
Ans. $(6y+\frac{1}{2})(6y-\frac{1}{2})$.

12. Resolve $(4x^2-\frac{1}{9})$ into factors. *Ans.* ——.

13. What are the factors of $(49a^2x^2-\frac{9}{16})$?
Ans. $(7ax+\frac{3}{4})(7ax-\frac{3}{4})$.

100. Binomials are occasionally found which are required to be resolved into *three* factors. In such cases we proceed as in **98.** We first find the monomial factor by inspection, and after dividing the whole quantity by this monomial, we proceed with the quotient just as in the preceding article.

Ex. 1. Resolve $(2cx^2-2cy^2)$ into three factors.

Analysis. By inspection we find that $2c$ is a factor common to both the terms; dividing $(2cx^2-2cy^2)$ by $2c$, we obtain (x^2-y^2) as the other factor. Our binomial can now be placed under the form of the two factors, $2c(x^2-y^2)$; then proceeding exactly as in the preceding article, we resolve (x^2-y^2) into the two factors $(x+y)$ and $(x-y)$; and now our whole binomial stands resolved into the three factors $2c(x+y)(x-y)$. *Ans.* $2c(x+y)(x-y)$.

EXAMPLES.

2. What are the three factors of $(4ac^2-9ay^2)$? *Ans.* $a(2c+3y)(2c-3y)$.

3. Resolve $(9xy^2-16m^2x)$ into three factors. *Ans.* ——.

4. What are the three factors of $(75b^2c-48cx^2)$? *Ans.* $3c(5b+4x)(5b-4x)$.

5. Resolve $(8cm^2-18cn^2)$ into three factors. *Ans.* ——.

6. What are the three factors of $(45bx^2-5b)$? *Ans.* $5b(3x+1)(3x-1)$.

7. Resolve $(63x^2y^2-7)$ into three factors. *Ans.* $7(3xy+1)(3xy-1)$.

8. What are the three factors of $(20a^2b^2-45)$? *Ans.* $5(2ab+3)(2ab-3)$.

SECTION VII.

PRIME AND COMPOSITE QUANTITIES.

101. ONE quantity is said to be a DIVISOR, or MEASURE, of another when the former is contained in the latter without a remainder. Thus, 6 is a *divisor* of 18; $3x$ is a divisor of $6ax$.

102. A PRIME quantity is one that can be divided only by itself and unity; 3, 5 and 7 are prime quantities; so are b, c and y.

103. A COMPOSITE quantity is one that can be divided by some integral quantity besides itself and unity. Thus, 15 is a composite quantity, because it can be divided by 3 and

5; bx is a composite quantity, because it can be divided by b and x.

A composite quantity is therefore the *product* of two or more *factors*, and therefore must be divisible by these factors.

104. A *prime* factor of a quantity is a prime quantity that will exactly divide it. Thus, 7 is a prime factor of 21; x is a prime factor of $5ax$.

GREATEST COMMON MEASURE.

105. Any quantity that will divide two or more quantities without a remainder is called their COMMON DIVISOR or COMMON MEASURE. Thus, 6 is a *common measure* of 12 and 18; $4ax$ is a *common measure* of $8a^2x$ and $12ax^2$.

EXAMPLES.

1. What is a common measure of 54 and 72? *Ans.* 9.

2. What is a common measure of $6ab$ and $9a^2b$? *Ans.* $3ab$.

3. What is a common measure of $10x^2y$ and $15xy^2$? *Ans.* ——.

4. What is a common measure of $14axy$ and $21amy$? *Ans.* $7ay$.

5. What is a common measure of $15abc$ and $20a^2b^2$? *Ans.* ——.

6. What is a common measure of $3bcx$, $6acx$ and $9b^2x^2$? *Ans.* $3x$.

106. The *greatest* quantity that will divide two or more quantities without a remainder is called their GREATEST COMMON MEASURE. Thus, 9 is the greatest common measure of 18 and 27; $5b$ is the greatest common measure of $10by$ and $15bx$.

EXAMPLES.

1. What is the greatest common measure of 24 and 36? *Ans.* 12.

2. What is the greatest common measure of $3a^2b$ and $6ab^2$? *Ans.* $3ab$.

3. What is the greatest common measure of $8ax$ and $4a^2x$? *Ans.* ——.

4. What is the greatest common measure of $15c^2d$ and $25cd^2$? *Ans.* $5cd$.

5. Give the greatest common measure of $14amy$ and $21a^2y^2$. *Ans.* ——.

6. Give the greatest common measure of $5ax$, $10a^2x$ and $15ax^2$. *Ans.* $5ax$.

107. It is obvious that two or more quantities may sometimes have several common measures, but they can have only *one* greatest common measure.

108. When the quantities are small, a little inspection will show us what is their greatest common measure, but when the quantities are larger or more complicated, we must resort to other methods.

Ex. 1. What is the greatest common measure of 120 and 252?

Analysis. Resolving 120 and 252 into their *prime* factors, we have

$$120 = 2 \times 2 \times 2 \times 3 \times 5,$$
$$252 = 2 \times 2 \times 3 \times 3 \times 7.$$

The *common* factors are 2, 2 and 3, and their product is the greatest common measure. *Ans.* 12.

In resolving 120 into its prime factors we found *three* 2's, and in resolving 252 we found only *two* 2's; hence, *one* of the 2's is not common to both numbers. In the same way, in resolving 120 we found only *one* 3, but in resolving 252 we found *two* 3's; hence, only *one* 3 was common to both numbers. Hence, we took only those factors that were *common* to both numbers, as only those could be common divisors.

What we have explained in reference to arithmetical quantities holds equally good with respect to algebraic quantities.

2. What is the greatest common measure of $27axy$ and $45ab^2x$?

Analysis. Resolving $27axy$ and $45ab^2x$ into their prime factors, we have

$$27axy = 3\times3\times3\times a\times x\times y,$$
$$45ab^2x = 3\times3\times5\times a\times b\times b\times x.$$

The *common* factors are 3, 3, a and x, and their product, $9ax$, is th greatest common measure. *Ans.* $9ax$.

From these illustrations we deduce the following

RULE.—*Resolve the quantities into their prime factors; then the product of all the prime factors* COMMON *to* ALL *the quantities will be the greatest common measure.*

EXAMPLES.

1. Find the greatest common measure of $21a^2b^2$ and $35abx$. *Ans.* $7ab$.
2. Find the G. C. M. of $36c^2dy$ and $48bcy^2$. *Ans.* $12cy$.
3. Find the G. C. M. of $28acx$, $35bc^2x$ and $49a^2cx^2$. *Ans.* $7cx$.
4. What is the G. C. M. of $27ab^2c$, $45a^2by$ and $54ab^2x$? *Ans.* $9ab$.
5. Find the G. C. M. of $24ax^2y$, $40bxy$ and $48cxy^2$. *Ans.* $8xy$.
6. What is the G. C. M. of $35xyz$, $45ay^2z$ and $50byz^2$? *Ans.* $5yz$.
7. Find the G. C. M. of $18a^2bx$, $24abx^2$ and $36ab^2x$. *Ans.* $6abx$.
8. What is the G. C. M. of $24a^2bc$, $36ab^2c$ and $48abc^2$? *Ans.* $12abc$.
9. Find the G. C. M. of $18am^2n^2$, $27bmn^2$ and $45cm^2n$. *Ans.* $9mn$.

10. Find the G. C. M. of (x^3-xy^2) and $(x^3-2x^2y+xy^2)$.

Analysis. Resolving x^3-xy^2 and $x^3-2x^2y+xy^2$ into their prime factors, we have

$$(x^3-xy^2)=x(x^2-y^2)=x(x+y)(x-y), \quad (100)$$

$$x^3-2x^2y+xy^2=x(x^2-2xy+y^2)=x(x-y)(x-y). \quad (98)$$

We observe that the *common* factors are x and $(x-y)$; hence, their product, $x(x-y)$ or (x^2-xy), is the greatest common measure.

Ans. (x^2-xy).

11. Find the G. C. M. of (a^2-x^2) and $(a^2+2ax+x^2)$.

Ans. $(a+x)$.

12. Find the G. C. M. of (b^2-y^2) and $(b^2-2by+y^2)$.

Ans. $(b-y)$.

13. Find the G. C. M. of (a^3-ax^2) and $(a^2-2ax+x^2)$.

Ans. $(a-x)$.

14. Find the G. C. M. of $(9x^2-1)$ and $(9x^2-6x+1)$.

Ans. $(3x-1)$.

15. Find the G. C. M. of $(4y^2-1)$ and $(4y^2+4y+1)$.

Ans. $(2y+1)$.

16. Find the G. C. M. of (a^2+ay) and $(a^2+2ay+y^2)$.

Ans. $(a+y)$.

COMMON MULTIPLE.

109. A MULTIPLE of a quantity is any quantity that can be divided by it without a remainder; thus, 21 is a multiple of 7, because it can be exactly divided by 7; $12ax$ is a multiple of $3a$, because it can be exactly divided by $3a$. Every quantity has an unlimited number of multiples.

Ex. 1. What is a multiple of $5x$? of $7y$? of $6ay$? of $7ab$?

2. What is a multiple of $9xy$? of $10a^2$? of $12b^2$? of $15ay$?

110. A COMMON MULTIPLE of two or more quantities is any quantity that can be divided by each of them without a remainder. Thus, 16 is a common multiple of 2, 4 and 8, because it can be exactly divided by each of those numbers;

$15ax$ is a common multiple of $3a$ and $5x$, because it can be exactly divided by each of those quantities.

Ex. 1. What is a common multiple of 16 and 24?
Ans. ——.

2. What is a common multiple of $3x$ and $4y$? *Ans.* ——.

3. What is a common multiple of $3b$ and $6c$? *Ans.* ——.

4. What is a common multiple of $3x$, $5x$ and $6x$?
Ans. ——.

111. The LEAST COMMON MULTIPLE of two or more quantities is the *least* quantity that can be divided by each of them without a remainder. Thus, 18 is the least common multiple of 3, 6 and 9, because it is the *least* number that can be divided by each of them without a remainder; $12a$ is the least common multiple of $3a$, $4a$ and $6a$; $30x$ is the least common multiple of $5x$, $6x$ and $10x$.

Ex. 1. What is the least common multiple of $4y$ and $6y$?
Ans. ——.

2. What is the least common multiple of $6x$, $8x$ and $12x$?
Ans. ——.

3. What is the least common multiple of $6y$, $10y$ and $15y$?
Ans. ——.

4. What is the least common multiple of $5a$, $8a$ and $10a$?
Ans. ——.

112. From the definition of a *multiple* given in **109**, it is obvious that a multiple of any quantity must be a *product*, and this product must be divisible by each of the factors of the given quantity. Thus, 30 is a multiple of 15. The factors of 15 are 3 and 5. Now, 3 and 5 will each divide 15 without a remainder; and so 3 and 5 will divide 30, which is a multiple of 15.

113. Upon the same principle, the *least common multiple* of two or more quantities is the *least product* that can be formed by the several prime factors that compose each quantity. Thus, 30 is the least common multiple of the three numbers, 6, 10 and 15, because it is the *least* number that will contain all three of the numbers without a remainder. Now, let us resolve each of these numbers into its prime factors: we have

$$6=2\times 3,$$
$$10=2\times 5,$$
$$15=3\times 5.$$

It will be noticed that the prime factor 2 occurs *twice;* 3 occurs *twice,* and 5 *twice.* Now, if we form a product by using *all* the prime factors, we shall obtain 900, which will be a common multiple, but not the *least* common multiple. But if we use each prime factor only *once,* we shall obtain the *least* possible product—viz. $2\times 3\times 5=30$—which is the *least* common multiple of the three given numbers.

From the foregoing explanations we derive the following

RULE. — *Resolve each of the quantities into its prime factors, and form a product by using the* SAME *prime factor but* ONCE. *The product thus formed will be the least common multiple required.*

NOTE.—For most practical purposes the least common multiple may be determined by a simple inspection.

EXAMPLES.

1. What is the least common multiple of 10, 15, 20 and 30?

Analysis.

$$10=2\times 5,$$
$$15=3\times 5,$$
$$20=2\times 2\times 5,$$
$$30=2\times 3\times 5.$$

Here we notice that the prime factor 2 is *common* to *three* of the numbers; we therefore use it as a factor *once;* 3 is common to *two* of the numbers, and we use it *once;* 5 is common to all the numbers, and we use it *once.* There is another factor, 2, which is *not* common to any of the other numbers, and therefore is a *separate* and *distinct* factor from the other 2's, although numerically equal in value; we therefore use this other 2 *once* as a factor. Our prime factors, therefore, are 2, 3, 5 and 2, and their product $2 \times 3 \times 5 \times 2 = 60$, which is the least common multiple of 10, 15, 20 and 30. *Ans.* 60.

2. Find the least common multiple of 21, 28 and 42? *Ans.* 84.

3. Find the L. C. M. of $3ab^2$, $6ab$ and $9a^2b$. *Ans.* $18a^2b^2$.

4. Find the L. C. M. of $12x$, $18x^2$ and $36x$. *Ans.* $36x^2$.

5. Find the L. C. M. of $5axy^2$, $10a^2xy$ and $15axy^2$. *Ans.* $30x^2xy^2$.

6. Find the L. C. M. of $9cx^2$, $12cx$ and $18c^2x$. *Ans.* $36c^2x^2$.

7. Find the L. C. M. of $4c^2xy$, $8cx^2y^2$ and $12c^2xy^2$. *Ans.* $24c^2x^2y^2$.

8. Find the L. C. M. of $9by$, $18b^2y$ and $27by^2$. *Ans.* $54b^2y^2$.

9. Find the L. C. M. of $12x^2$, $16x$ and $24x^2$. *Ans.* $48x^2$.

10. Find the L. C. M. of $18y^2$, $24y$ and $36y^2$. *Ans.* $72y^2$.

11. Find the L. C. M. of $7ax^2$, $14a^2x$ and $21ax$. *Ans.* $42a^2x^2$.

12. Find the L. C. M. of $3(c-x)$, $2(c+x)$ and $6(c^2-x^2)$. *Ans.* $6(c^2-x^2)$.

13. Find the L. C. M. of $5(m+n)$, $10(m-n)$ and $3(m^2-n^2)$. *Ans.* $30(m^2-n^2)$.

14. Find the L. C. M. of $3x(c+d)$, $5x^2(c-d)$ and $10x(c^2-d^2)$. *Ans.* $30x^2(c^2-d^2)$.

15. Find the L. C. M. of (m^2+y^2), $(m+y)$ and m^2-y^2. *Ans.* m^4-y^4.

16. Find the L. C. M. of $16(a+b)$, $24(a-b)$ and (a^2-b^2). *Ans.* $48(a^2-b^2)$.

SECTION VIII.

FRACTIONS.

114. WHEN a number or quantity is divided into equal parts, these equal parts are called FRACTIONS;* hence, a FRACTION is an expression representing a part or parts of a unit or whole quantity. Thus, *one-fifth* represents a *part* of a unit; *four-fifths* represent *parts* of a unit. Algebraic fractions, for all practical purposes, are similar to fractions in Arithmetic, and the rules and operations applicable to arithmetical fractions apply equally to fractions in Algebra.

115. Fractions originate in division. If all quantities were capable of being divided without leaving a remainder, there would be no fractions.

116. A fraction is commonly represented by two quantities, one placed above the other, with a line between them; thus, $\frac{5}{7}$ represents a fraction, and indicates that 5 is divided by 7; $\frac{a}{b}$ represents an algebraic fraction, and indicates that the quantity represented by a is divided by the quantity represented by b. If $a = 4$ and $b = 9$, then $\frac{a}{b} = \frac{4}{9}$, and so on.

The quantity *below* the line is called the DENOMINATOR; the quantity *above* the line is called the NUMERATOR.

117. The DENOMINATOR† shows into how many parts the unit or quantity is divided. In the fraction $\frac{1}{5}$ the unit is supposed to be divided into *five* equal parts; in $\frac{1}{n}$ it is divided into n parts.

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* *Fraction*, from the Latin *frango, to break.*

† *Denominator*, from the Latin word *denomino*, signifying *to name*, because it gives *name* or *denomination* to the fraction.

The NUMERATOR [*] shows how many of the equal parts are used. Thus, in the fraction $\frac{3}{5}$ *three* parts are used; in $\frac{b}{n}$, b parts are used.

118. The numerator and the denominator, taken together, are called the TERMS of the fraction. Thus, $\frac{m}{n}$ is a fraction: m and n are the terms; n is the denominator, and shows that the unit or quantity is divided into n equal parts, making each part $\frac{1}{n}$; m is the numerator, and shows that m of the equal parts are used.

119. The VALUE of a fraction is the *quotient* of the numerator divided by the denominator. The value of $\frac{15}{5}$ is 3; of $\frac{ax}{a}$ is x; of $\frac{mp}{p}$ is m.

Hence, we see that the numerator of a fraction is a *dividend;* the denominator is a *divisor;* and the value of the fraction is a *quotient.* All the rules of division, therefore, are applicable to fractions.

120. An INTEGER,[†] or *entire* quantity, is one that is not expressed under a fractional form, as 5, $3b$, $2x$, $4y$, etc.

121. A MIXED quantity consists of an integer and a fraction, as $2\frac{3}{4}$, $(m+\frac{a}{3})$, etc.

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[*] *Numerator,* from the Latin *numero, to number,* because it *numbers* the parts, or tells *how many* there are.

[†] *Integer* is a Latin word, meaning *whole* or *entire.*

122. Any integral quantity may be expressed as a fraction by writing 1 below it for a denominator. Thus, 5 may be written $\frac{5}{1}$; b may be written $\frac{b}{1}$, etc.

123. A COMPOUND fraction is a fraction of a fraction, or several fractions connected by the word *of;* as, $\frac{2}{3}$ of $\frac{4}{5}$, $\frac{3}{4}$ of $\frac{a}{b}$, etc.

A COMPLEX fraction is one which has a fraction either in its numerator or in its denominator, or in both; as, $\frac{3\frac{1}{4}}{5}$, $\frac{5}{2\frac{1}{4}}$, $\frac{a+\frac{b}{c}}{r-\frac{m}{n}}$, etc.

In dealing with fractions there are certain fundamental principles which must always be kept in mind. To these we now call the attention of the student.

124. If we take the fraction $\frac{2}{9}$, and multiply the numerator by 3, we shall have $\frac{6}{9}$. Now, 6 *ninths* is evidently *three* times as much as 2 *ninths;* hence we derive the following Principles:

PRINCIPLE I. *Multiplying the numerator of a fraction by any integral quantity* MULTIPLIES *the value of the fraction.*

Again: if we divide the numerator of $\frac{6}{9}$ by 3, we have $\frac{2}{9}$. Now, 2 *ninths* is evidently *one third of* 6 *ninths;* hence, we derive the following

PRINCIPLE II. *Dividing the numerator of a fraction by any quantity* DIVIDES *the value of the fraction.*

Again: if we multiply the denominator of $\frac{2}{3}$ by 3, we have $\frac{2}{9}$ But 2 *ninths* is only *one third* as much as 2 *thirds,* because it requires 3 *ninths* to make 1 *third;* hence, Principle III. *Multiplying the denominator of a fruction by any quantity* DIVIDES *the value of the fraction.*

Again: if we divide the denominator of $\frac{2}{9}$ by 3, we have $\frac{2}{3}$. But 2 *thirds* is 3 times as much as 2 *ninths*, since it requires 3 *ninths* to make 1 *third;* hence,

Principle IV. *Dividing the denominator of a fraction by any quantity* MULTIPLIES *the value of the fraction.*

125. According to the Principles above stated, it is evident that multiplying the numerator has the same effect as dividing the denominator, and dividing the numerator has the same effect as multiplying the denominator. Hence,

Principle V. *If both the numerator and the denominator be multiplied or divided by the same quantity, the value of the fraction will not be altered.*

SIGNS OF FRACTIONS.

126. One of the most troublesome things to a beginner in algebraic fractions is the management of the positive and negative signs. The effect of changes in these signs should be made perfectly clear at the outset, as this will save the student much labor and perplexity in the future.

The sign prefixed to the dividing-line of a fraction shows what operation is to be performed on the *whole* value of the fraction; thus, $2a+\frac{3b-4c+2d}{5}$ shows that the *whole* fraction, $\frac{3b-4c+2d}{5}$, is to be *added to* $2a$; but $2a-\frac{3b-4c+2d}{5}$ denotes that the *whole* value of the fraction, $\frac{3b-4c+2d}{5}$, is to be *subtracted* from $2a$.

127. The mixed quantity $x+\frac{ac}{c}=x+a$,

$$\text{but } x-\frac{ac}{c}=x-a.$$

$$\text{Again: } b-\frac{am}{m}=b-a,$$

$$\text{but } b+\frac{am}{m}=b+a.$$

From this we see—

That changing the sign before the dividing-line of a fraction has the effect of changing the whole value of the fraction from positive to negative, or from negative to positive.

Again: The fraction $\frac{ax-bx+cx}{x}=a-b+c$,

$$\text{but } \frac{-ax+bx-cx}{x}=-a+b-c.$$

Hence it follows—

That if we change all the signs of the numerator, the value of the fraction will be changed from positive to negative, or from negative to positive.

Again: The fraction $\frac{ax-bx+cx}{x}=a-b+c$,

$$\text{but } \frac{ax-bx+cx}{-x}=-a+b-c.$$

Hence we learn—

That if we change the signs of the denominator, the value of the fraction will be changed from positive to negative, or from negative to positive.

All that we have said may be summed up as follows:

If the sign prefixed to a fraction, or all the signs of the numerator, or all the signs of the denominator, be changed, the value of the fraction will be changed from positive to negative, or from negative to positive.

128. The mixed quantity $x+\frac{bm}{m}=x+b$,

$$\text{and, } x-\left(\frac{-bm}{m}\right)=x-(-b)=x+b, \qquad (1)$$

$$\text{and, } x-\left(\frac{bm}{-m}\right)=x-(-b)=x+b, \qquad (2)$$

$$\text{and, } x+\left(\frac{-bm}{-m}\right)=x+(+b)=x+b. \qquad (3)$$

It will be observed that in the formula (1) we changed the sign before the fraction, and also the sign of the numerator; in formula (2) we changed the sign before the fraction and the sign of the denominator; and in formula (3) we changed the sign of the numerator and that of the denominator; and yet the value of the fraction *was not altered.* Hence, *if any two changes be made at the same time, the value of the fraction will not be altered.*

REDUCTION OF FRACTIONS.

129. REDUCTION* OF FRACTIONS consists in changing them from one *form* to another without altering their *value.*

CASE I.

130. To Reduce a Fraction to its Lowest Terms.

NOTE.—A fraction is reduced to its *lowest terms* when its numerator and its denominator have no *common factor.*

Ex. 1. Reduce $\frac{8}{20}$ to its lowest terms.

Analysis. Dividing both terms of the fraction by 2, we have $\frac{4}{10}$; again dividing by 2, we obtain $\frac{2}{5}$, whose terms have no common factor.

OPERATION.

$$\frac{2)8}{2)20}=\frac{4}{10}$$

$$\frac{2)4}{2)10}=\frac{2}{5}$$

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* *Reduction,* from the Latin *reduco, to bring back, to bring down.*

OPERATION.

Or, we may simplify the operation thus: $\frac{4)8}{4)20}=\frac{2}{5}$

If we divide both terms of the fraction by 4, their *greatest common divisor*, the fraction will at once be reduced to its lowest terms.

2. Reduce $\frac{10ac}{15ax}$ to its lowest terms.

Analysis. We see at a glance that $5a$ is a factor common to both numerator and denominator, and we divide by this factor and obtain $\frac{2c}{3x}$.

OPERATION.

$\frac{5a)10ac}{5a)15ax}=\frac{2c}{3x}$

From the foregoing explanations we deduce the following

RULE.—*Divide both numerator and denominator by any quantity that will divide them both without a remainder;*

Or,

Divide them at once by their greatest common divisor.

EXAMPLES.

3. Reduce $\frac{10abx}{25aby}$ to its lowest terms. *Ans.* $\frac{2x}{5y}$.

4. Reduce $\frac{20a^2bc}{24abd}$ to its lowest terms. *Ans.* $\frac{5ac}{6d}$.

5. Reduce $\frac{28a^2x}{35ax^2}$ to its lowest terms. *Ans.* $\frac{4a}{5x}$.

6. Reduce $\frac{27a^2bx}{36a^3b}$ to its lowest terms. *Ans.* $\frac{3x}{4a}$.

7. Reduce $\frac{40ax^2}{45ax}$ to its lowest terms. *Ans.* $\frac{8x}{9}$.

8. Reduce $\frac{54a^2bx}{60ax}$ to its lowest terms. *Ans.* $\frac{9ab}{10}$.

9. Reduce $\frac{72ay}{84a^2b}$ to its lowest terms. *Ans.* $\frac{6y}{7ab}$.

10. Reduce $\frac{90a^3c}{108a^2x}$ to its lowest terms. *Ans.* $\frac{5ac}{6x}$.

11. Reduce $\frac{120x^2}{144x^3}$ to its lowest terms. *Ans.* $\frac{5}{6x}$.

12. Reduce $\frac{a^2+ax}{ab+ay}$ to its lowest terms. *Ans.* $\frac{a+x}{b+y}$.

13. Reduce $\frac{x^2+xy}{bx+cx}$ to its lowest terms. *Ans.* $\frac{x+y}{b+c}$

14. Reduce $\frac{ac+2c}{bc+3c}$ to its lowest terms. *Ans.* $\frac{a+2}{b+3}$

15. Reduce $\frac{a^2-a}{ab+ac}$ to its lowest terms. *Ans.* $\frac{a-1}{b+c}$

16. Reduce $\frac{cx+x^2}{x^2-x}$ to its lowest terms. *Ans.* $\frac{c+x}{x-1}$

17. Reduce $\frac{ax+ay}{bx+by}$ to its lowest terms. *Ans.* $\frac{a}{b}$.

18. Reduce $\frac{a^2-b^2}{a^2+2ab+b^2}$ to its lowest terms. *Ans.* $\frac{a-b}{a+b}$

19. Reduce $\frac{a^2+2ax+x^2}{a^2-x^2}$ to its lowest terms. *Ans.* $\frac{a+x}{a-x}$

20. Reduce $\frac{10a^3+15a^2b+20a}{15a^2}$ to its lowest terms.
Ans. $\frac{2a^2+3ab+4}{3a}$

21. Reduce $\frac{16x^2-24x^3}{8x^3}$ to its lowest terms. *Ans.* $\frac{2-3x}{x}$.

22. Reduce $\frac{39y^3-13y^2+26y}{13y^2}$ to its lowest terms.
Ans. $\frac{3y^2-y+2}{y}$

23. Reduce $\frac{a^2x+ax^2}{abx-axy}$ to its lowest terms. *Ans.* $\frac{a+x}{b-y}$

24. Reduce $\frac{7a^2y-21ay}{14ay+28a^2y}$ to its lowest terms. *Ans.* $\frac{a-3}{2+4a}$

25. Reduce $\frac{3(a^2-b^2)}{5(a^2+2ab+b^2)}$ to its lowest terms.

Ans. $\frac{3(a-b)}{5(a+b)}$.

CASE II.

131. To Reduce a Fraction to an Entire or Mixed Quantity.

Ex. 1. Reduce $\frac{24}{7}$ to a mixed quantity.

Analysis. Since 7 *sevenths* make 1 *whole one*, 24 *sevenths* will make as many *whole ones* as 7 is contained times in 24, which will be 3 *times* and 3 *sevenths*.

OPERATION.

$7\overline{)24}$

$3\frac{3}{7}$

Ans. $3\frac{3}{7}$.

Hence, we derive the following

RULE. *Divide the numerator by the denominator; if there be a remainder, place it over the denominator.*

EXAMPLES.

2. Reduce $\frac{45}{8}$ to a mixed quantity. *Ans.* $5\frac{5}{8}$.

3. Reduce $\frac{ax+b}{x}$ to a mixed quantity. *Ans.* $a+\frac{b}{x}$.

4. Reduce $\frac{y^2+ay-x}{y}$ to a mixed quantity.

Ans. $y+a-\frac{x}{y}$.

5. Reduce $\frac{6x^2-3x+2a}{3x}$ to a mixed quantity.

Ans. $2x-1+\frac{2a}{3x}$.

6. Reduce $\frac{20a^2-15a+3y}{5a}$ to a mixed quantity.

Ans. $4a-3+\frac{3y}{5a}$.

7. Reduce $\frac{12x^3-16x^2+2a}{4x^2}$ to a mixed quantity.

Ans. $3x-4+\frac{2a}{4x^2}$.

8. Reduce $\frac{14y^2-21y^3+3x}{7y^2}$ to a mixed quantity.
Ans. $2-3y+\frac{3x}{7y^2}$.

9. Reduce $\frac{a^2-2ax+x^2}{a-x}$ to an entire quantity. *Ans.* $a-x$.

10. Reduce $\frac{x^2-y^2}{x+y}$ to an entire quantity. *Ans.* $x-y$.

11. Reduce $\frac{21a^3+35a^2-5bc}{7a^2}$ to a mixed quantity.
Ans. $3a+5-\frac{5bc}{7a^2}$.

12. Reduce $\frac{24b^2-32b+3a}{8b}$ to a mixed quantity.
Ans. $3b-4+\frac{3a}{8b}$.

13. Reduce $\frac{a^2-x^2+5b}{a+x}$ to a mixed quantity.
Ans. $a-x+\frac{5b}{a+x}$

14. Reduce $\frac{b^2-y^2-4a}{b-y}$ to a mixed quantity.
Ans. $b+y-\frac{4a}{b-y}$.

15. Reduce $\frac{x^2+2xy+y^2-5a}{x+y}$ to a mixed quantity.
Ans. $x+y-\frac{5a}{x+y}$.

16. Reduce $\frac{x^3-y^3}{x-y}$ to an entire quantity. *Ans.* ——

CASE III.

132. To Reduce a Mixed Quantity to a Fractional Form.

Ex. 1. Reduce $4\frac{3}{5}$ to the form of a fraction.

Analysis. In 1 unit there are 5 *fifths*, and in 4 units there are 4 times as many *fifths* = 20 *fifths;* and 3 more *fifths* make 23 *fifths*, or $\frac{23}{5}$. *Ans.* $\frac{23}{5}$.

2. Reduce $a+\frac{b}{c}$ to a fractional form.

Analysis. 1 unit is equal to $\frac{c}{c}$, and a units will be a times as much $=\frac{ac}{c}$; and $\frac{b}{c}$ units more make $\frac{ac}{c}+\frac{b}{c}=\frac{ac+b}{c}$ units. *Ans.* $\frac{ac+b}{c}$.

From the Analysis we derive the following

RULE.—*Multiply the integral quantity by the denominator; to the product annex the numerator with its proper sign, and place the result over the given denominator.*

EXAMPLES.

3. Reduce $9\frac{5}{7}$ to a fractional form. *Ans.* $\frac{68}{7}$.

4. Reduce $2x+\frac{3a}{5}$ to a fractional form. *Ans.* $\frac{10x+3a}{5}$.

5. Reduce $5y-\frac{3b}{7}$ to a fractional form. *Ans.* $\frac{35y-3b}{7}$.

6. Reduce $x+\frac{2a+b}{4}$ to a fractional form.
Ans. $\frac{4x+2a+b}{4}$.

7. Reduce $4a+\frac{2x-3}{5}$ to a fractional form.
Ans. $\frac{20a+2x-3}{5}$.

8. Reduce $3y+\frac{4a-5}{4}$ to a fractional form.
Ans. $\frac{12y+4a-5}{4}$.

9. Reduce $x+\frac{b-x^2}{x}$ to a fractional form. *Ans.* $\frac{b}{x}$.

10. Reduce $4a+\frac{5c-x}{7x}$ to a fractional form.
Ans. $\frac{28ax+5c-x}{7x}$.

11. Reduce $3x+\frac{4a+2x^2}{4x}$ to a fractional form.

Ans $\frac{14x^2+4a}{4x}$.

12. Reduce $5y+\frac{2b-3y^2}{3y}$ to a fractional form.

Ans. $\frac{12y^2+2b}{3y}$.

13. Reduce $3a-\frac{2x+3}{3}$ to a fractional form.

Ans. $\frac{9a-2x-3}{3}$.

The − sign before the fraction $\frac{2x+3}{3}$ indicates that the *whole* of the numerator is to be *subtracted*, which is done by *changing all the signs* of the numerator. This is a point which students are prone to overlook.

14. Reduce $5x-\frac{3a+6}{2}$ to a fractional form.

Ans. $\frac{10x-3a-6}{2}$.

15. Reduce $4a-\frac{2x-5}{4}$ to a fractional form.

Ans. $\frac{16a-2x+5}{4}$.

16. Reduce $3a^2-\frac{4x+6}{5}$ to a fractional form.

Ans. $\frac{15a^2-4x-6}{5}$.

17. Reduce $5y-\frac{3b-4}{7}$ to a fractional form.

Ans. $\frac{35y-3b+4}{7}$.

18. Reduce $4a-\frac{3x+2y-5}{3}$ to a fractional form.

Ans. $\frac{12a-3x-2y+5}{3}$.

19. Reduce $3x-\frac{5y-2a+3}{5}$ to a fractional form.
Ans. $\frac{15x-5y+2a-3}{5}$.

20. Reduce $4x^2+\frac{3a-2x^2+4}{3}$ to a fractional form.
Ans. $\frac{10x^2+3a+4}{3}$.

21. Reduce $4a+\frac{2x-5}{a-b}$ to a fractional form.
Ans. $\frac{4a^2-4ab+2x-5}{a-b}$.

22. Reduce $5+\frac{3x-4}{4x}$ to a fractional form. Ans. $\frac{23x-4}{4x}$.

23. Reduce $1+3x+\frac{x-5}{2x}$ to a fractional form.
Ans. $\frac{3x+6x^2-5}{2x}$.

24. Reduce $y-x-\frac{y^2-xy}{x}$ to a fractional form.
Ans. $\frac{2xy-x^2-y^2}{x}$.

25. Reduce $a+b+\frac{2a^2-2b^2}{a-b}$ to a fractional form.
Ans. $\frac{3a^2-3b^2}{a-b}$.

CASE IV.

133. To Reduce a Compound Fraction to a Simple One.

Ex. 1. Reduce $\frac{1}{3}$ of $\frac{1}{5}$ to a simple fraction.

Analysis. $\frac{1}{3}$ of $\frac{1}{5}$ is $\frac{1}{15}$. We know this to be true, because if we cut an orange into 5 equal parts, each part will be $\frac{1}{5}$ of the whole orange. Now, if we cut each of these *fifths* into *three* equal parts, we see that the whole orange will be divided into *fifteen* equal parts, making each part $\frac{1}{15}$ of the whole orange; therefore, $\frac{1}{3}$ of $\frac{1}{5}$ is $\frac{1}{15}$. *Ans.* $\frac{1}{15}$.

OPERATION.
$\frac{1}{3}$ of $\frac{1}{5}=\frac{1}{15}$

2. Reduce $\frac{3}{5}$ of $\frac{4}{7}$ to a simple fraction.

Analysis. $\frac{1}{5}$ of $\frac{1}{7}=\frac{1}{35}$; then $\frac{3}{5}$ of $\frac{1}{7}$ will be 3 times as much, which will be $\frac{1}{35}$; and $\frac{4}{5}$ of $\frac{3}{7}$ will be 4 times as much, which will be $\frac{12}{35}$.

OPERATION.

$\frac{4}{5}$ of $\frac{3}{7} = \frac{4 \times 3}{5 \times 7} = \frac{12}{35}$

By observing the Analysis we see the reason for the following

RULE.—***Multiply the numerators together for a new numerator, and the denominators together for a new denominator.***

NOTE.—Mixed quantities should first be reduced to fractional forms.

EXAMPLES.

3. Reduce $\frac{2}{3}$ of $\frac{5}{7}$ to a simple fraction. *Ans.* $\frac{10}{21}$.

4. Reduce $\frac{4}{9}$ of $\frac{4x}{5}$ to a simple fraction. *Ans.* $\frac{16x}{45}$.

5. Reduce $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{3}{7}$ to a simple fraction. *Ans.* $\frac{8}{35}$.

6. Reduce $\frac{3}{4}$ of $\frac{5}{7}$ of $\frac{3c}{5}$ to a simple fraction. *Ans.* $\frac{9c}{28}$.

CASE V.

134. To Reduce Fractions to a Common Denominator.

NOTE.—By *common* denominator is meant the same denominator.

Ex. 1. Reduce $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{4}{5}$ to a common denominator.

The denominators all multiplied together give 60 for a product; and therefore we may reduce these fractions to *sixtieths*, as follows:

Analysis. One unit is equal to $\frac{60}{60}$; then $\frac{1}{3}$ of a unit will be $\frac{20}{60}$, and $\frac{2}{3}$ will be $\frac{40}{60}$. Again: Since 1 unit $= \frac{60}{60}$, $\frac{1}{4}$ of a unit is $\frac{15}{60}$, and $\frac{3}{4}$ will be $\frac{45}{60}$. Lastly, since 1 unit $= \frac{60}{60}$, then $\frac{1}{5}$ of a unit is $\frac{12}{60}$, and $\frac{4}{5}$ will be $\frac{48}{60}$. Hence, we see that $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{4}{5}$, when reduced, become respectively $\frac{40}{60}$, $\frac{45}{60}$ and $\frac{48}{60}$.

Ans. $\frac{40}{60}$, $\frac{45}{60}$, $\frac{48}{60}$.

OPERATION.

$\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$

$3 \times 4 \times 5 = 60$

$1 = \frac{60}{60}$, $1 = \frac{60}{60}$, $1 = \frac{60}{60}$

$\frac{1}{3} = \frac{20}{60}$, $\frac{1}{4} = \frac{15}{60}$, $\frac{1}{5} = \frac{12}{60}$

$\frac{2}{3} = \frac{40}{60}$, $\frac{3}{4} = \frac{45}{60}$, $\frac{4}{5} = \frac{48}{60}$

Hence, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5} = \frac{40}{60}$, $\frac{45}{60}$, $\frac{48}{60}$

The operation therefore virtually consists in multiplying both terms of each fraction by the product of all the denominators except its own.

2. Reduce $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ to a common denominator.

The product of the denominators is bdf; hence, we may reduce these fractions to others whose *common* denominator shall be bdf.

Analysis. 1 unit is equal to $\frac{bdf}{bdf}$; then $\frac{1}{b}=\frac{df}{bdf}$, and $\frac{a}{b}$ will equal a times as much $=\frac{adf}{bdf}$. Again: Since 1 unit $=\frac{bdf}{bdf}$, then $\frac{1}{d}=\frac{bf}{bdf}$, and $\frac{c}{d}$ will be c times as much $=\frac{bcf}{bdf}$. Lastly, since 1 unit $=\frac{bdf}{bdf}$, then $\frac{1}{f}=\frac{bd}{bdf}$, and $\frac{e}{f}$ will be e times as much $=\frac{bde}{bdf}$. Hence, we see that $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$, when reduced, become $\frac{adf}{bdf}$, $\frac{bcf}{bdf}$ and $\frac{bde}{bdf}$.

OPERATION.

$$\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$$

$$b\times d\times f=bdf$$

$$1=\frac{bdf}{bdf}, 1=\frac{bdf}{bdf}, 1=\frac{bdf}{bdf}$$

$$\frac{1}{b}=\frac{df}{bdf}, \frac{1}{d}=\frac{bf}{bdf}, \frac{1}{f}=\frac{bd}{bdf}$$

$$\frac{a}{b}=\frac{adf}{bdf}, \frac{c}{d}=\frac{bcf}{bdf}, \frac{e}{f}=\frac{bde}{bdf}$$

Hence, $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}=\frac{adf}{bdf}, \frac{bcf}{bdf}, \frac{bde}{bdf}$

Ans. $\frac{adf}{bdf}, \frac{bcf}{bdf}, \frac{bde}{bdf}$.

From the foregoing illustrations we deduce the following

RULE.—***Multiply both terms of each fraction by the product of all the denominators except its own.***

NOTE I.—Mixed quantities must be reduced to a fractional form, and compound fractions to simple ones.

NOTE II.—*To Teachers.* It would be advantageous for the student to work *many*, if not all, the examples under this and the following Case by *Analysis*, as mental discipline will thereby be promoted.

EXAMPLES.

Reduce the following fractions to others having a common denominator:

3. Reduce $\frac{4}{5}$, $\frac{3}{4}$ and $\frac{5}{7}$. *Ans.* $\frac{112}{140}$, $\frac{105}{140}$, $\frac{100}{140}$.

4. Reduce $\frac{3a}{4}$, $\frac{5x}{6}$ and $\frac{b}{2}$. *Ans.* $\frac{36a}{48}$, $\frac{40x}{48}$, $\frac{24b}{48}$.

5. Reduce $\frac{x}{2}, \frac{5x}{7}$ and $\frac{2x}{3}$. *Ans.* $\frac{21x}{42}, \frac{30x}{42}, \frac{28x}{42}$.

6. Reduce $\frac{2a}{3}, \frac{3a}{4}$ and $\frac{5a}{6}$. *Ans.* $\frac{48a}{72}, \frac{54a}{72}, \frac{60a}{72}$.

7. Reduce $\frac{a}{3}, \frac{2b}{x}$ and $\frac{3c}{y}$. *Ans.* $\frac{axy}{3xy}, \frac{6by}{3xy}, \frac{9cx}{3xy}$.

8. Reduce $\frac{x}{4}, \frac{3c}{y}$ and $\frac{5a}{b-x}$.

Ans. $\frac{bxy-x^2y}{4by-4xy}, \frac{12bc-12cx}{4by-4xy}, \frac{20ay}{4by-4xy}$

9. Reduce $\frac{x+2}{3}$ and $\frac{y-5}{4b}$. *Ans.* $\frac{4bx+8b}{12b}, \frac{3y-15}{12b}$.

10. Reduce $\frac{x+y}{4a}$ and $\frac{x-y}{3b}$. *Ans.* $\frac{3bx+3by}{12ab}, \frac{4ax-4ay}{12ab.}$.

11. Reduce $\frac{2x}{5}, \frac{4x}{7}$ and $\frac{x}{2}$. *Ans.* $\frac{28x}{70}, \frac{40x}{70}, \frac{35x}{70}$.

12. Reduce $\frac{4}{5}, \frac{2x}{3}$ and $\frac{b+3x}{4}$. *Ans.* $\frac{48}{60}, \frac{40x}{60}, \frac{15b+45x}{60}$.

CASE VI.

135. To Reduce Fractions to their LEAST Common Denominator.

By the method just explained fractions may be reduced to a *common* denominator, but it frequently happens that in such cases the denominator thus obtained will not be the *least common* denominator. For convenience of calculation it is frequently desirable to make the denominators as small as possible, which is done by reducing the fractions to their *least common* denominator.

Ex. 1. Reduce $\frac{3}{5}$, $\frac{5}{6}$ and $\frac{7}{10}$ to their least common denominator.

Analysis. A little inspection of these denominators will show us

OPERATION.

$$\tfrac{3}{5},\ \tfrac{5}{6},\ \tfrac{7}{10}$$

$$1=\tfrac{30}{30},\ 1=\tfrac{30}{30},\ 1=\tfrac{30}{30}$$

$$\tfrac{1}{5}=\tfrac{6}{30},\ \tfrac{1}{6}=\tfrac{5}{30},\ \tfrac{1}{10}=\tfrac{3}{30}$$

$$\tfrac{3}{5}=\tfrac{18}{30},\ \tfrac{5}{6}=\tfrac{25}{30},\ \tfrac{7}{10}=\tfrac{21}{30}$$

Hence, $\tfrac{3}{5}$, $\tfrac{5}{6}$, $\tfrac{7}{10}=\tfrac{18}{30}$, $\tfrac{25}{30}$, $\tfrac{21}{30}$

that the *least* number in which they can all be contained is 30; hence, 30 is their *least common* denominator. The question then is, to reduce $\tfrac{3}{5}$, $\tfrac{5}{6}$ and $\tfrac{7}{10}$ to *thirtieths*. We say: 1 unit $=\tfrac{30}{30}$; then $\tfrac{1}{5}$ of a unit will be equal to $\tfrac{6}{30}$ and $\tfrac{3}{5}$ will be $\tfrac{18}{30}$. Again: Since 1 unit $=\tfrac{30}{30}$, $\tfrac{1}{6}$ of a unit will be equal to $\tfrac{5}{30}$ and $\tfrac{5}{6}$ will be $\tfrac{25}{30}$. Lastly: Since 1 unit $=\tfrac{30}{30}$, then $\tfrac{1}{10}$ of a unit $=\tfrac{3}{30}$, and $\tfrac{7}{10}$ will be $\tfrac{21}{30}$. Hence, $\tfrac{3}{5}$, $\tfrac{5}{6}$ and $\tfrac{7}{10}$, when reduced, become respectively $\tfrac{18}{30}$, $\tfrac{25}{30}$ and $\tfrac{21}{30}$.

Ans. $\tfrac{18}{30}$, $\tfrac{25}{30}$, $\tfrac{21}{30}$.

Hence, in practice, we employ the following

RULE.—*Find the* LEAST *common multiple of all the denominators for the least common denominator. Divide this least common denominator by the denominator of each fraction, and multiply the quotient by its numerator; the several products will be the numerators required.*

NOTE.—Mixed quantities should be reduced to fractional forms, and each fraction to its lowest terms, before applying this rule.

EXAMPLES.

Reduce the following to equivalent fractions having the *least* common denominator:

2. Reduce $\tfrac{5}{9}$, $\tfrac{7}{12}$ and $\tfrac{5}{18}$. *Ans.* $\tfrac{20}{36}$, $\tfrac{21}{36}$, $\tfrac{10}{36}$.

3. Reduce $\dfrac{x}{4}$, $\dfrac{5x}{6}$ and $\dfrac{2x}{3}$. *Ans.* $\dfrac{3x}{12}$, $\dfrac{10x}{12}$, $\dfrac{8x}{12}$.

4. Reduce $\dfrac{a}{3b}$, $\dfrac{c}{4x}$ and $\dfrac{5d}{8}$. *Ans.* $\dfrac{8ax}{24bx}$, $\dfrac{6bc}{24bx}$, $\dfrac{15bdx}{24bx}$.

5. Reduce $\dfrac{2b}{3m}$, $\dfrac{5a}{6m^2}$ and $\dfrac{3x}{4}$. *Ans.* $\dfrac{8bm}{12m^2}$, $\dfrac{10a}{12m^2}$, $\dfrac{9m^2x}{12m^2}$.

6. Reduce $\dfrac{3a}{4b^3}$, $\dfrac{4c}{5b^2}$ and $\dfrac{7x}{10b^4}$. *Ans.* $\dfrac{15ab}{20b^4}$, $\dfrac{16b^2c}{20b^4}$, $\dfrac{14x}{20b^4}$.

7. Reduce $\frac{a-x}{5}, \frac{9y}{10}$ and $\frac{5a}{6c}$. *Ans.* $\frac{6ac-6cx}{30c}, \frac{27cy}{30c}, \frac{25a}{30c}$.

8. Reduce $\frac{2a}{9n}, \frac{4b}{6mn}$ and $\frac{2x}{3m}$.

Ans. $\frac{4am}{18mn}, \frac{12b}{18mn}, \frac{12nx}{18mn}$.

9. Reduce $\frac{a+x}{3c}, \frac{4a}{7d}$ and $\frac{a-x}{14cd}$.

Ans. $\frac{14ad+14dx}{42cd}, \frac{24ac}{42cd}, \frac{3a-3x}{42cd}$.

10. Reduce $\frac{2x}{3}, \frac{5c}{6}$ and $\frac{1+x}{1-x}$.

Ans. $\frac{4x-4x^2}{6-6x}, \frac{5c-5cx}{6-6x}, \frac{6+6x}{6-6x}$.

11. Reduce $\frac{1+x}{6c}, \frac{3a}{4b}$ and $\frac{a+b}{3bc}$.

Ans. $\frac{2b+2bx}{12bc}, \frac{9ac}{12bc}, \frac{4a+4b}{12bc}$.

12. Reduce $\frac{4x}{5}, \frac{a+x}{9}$ and $\frac{7x}{15}$. *Ans.* $\frac{36x}{45}, \frac{5a+5x}{45}, \frac{21x}{45}$.

13. Reduce $\frac{5}{a-x}, \frac{3}{a+x}$ and $\frac{3a}{a^2-x^2}$.

Ans. $\frac{5a+5x}{a^2-x^2}, \frac{3a-3x}{a^2-x^2}, \frac{3a}{a^2-x^2}$.

14. Reduce $\frac{1}{1+x}, \frac{4}{1-x^2}$ and $\frac{5c}{1-x}$.

Ans. $\frac{1-x}{1-x^2}, \frac{4}{1-x^2}, \frac{5c+5cx}{1-x^2}$.

15. Reduce $\frac{5x}{7}, \frac{9x}{14}$ and $\frac{a-x}{10}$. *Ans.* $\frac{50x}{70}, \frac{45x}{70}, \frac{7a-7x}{70}$.

16. Reduce $\frac{5x}{12a}$, $\frac{7a}{8b}$ and $a+\frac{3c}{4}$.

Ans. $\frac{10bx}{24ab}$, $\frac{21a^2}{24ab}$, $\frac{24a^2b+18abc}{24ab}$.

17. Reduce $\frac{3x}{7}$, $\frac{5x}{14}$ and $\frac{8x}{21}$. *Ans.* $\frac{18x}{42}$, $\frac{15x}{42}$, $\frac{16x}{42}$.

ADDITION OF FRACTIONS.

CASE I.

136. When the Denominators are Alike.

Ex. 1. What is the sum of $\frac{4}{9}$, $\frac{7}{9}$ and $\frac{8}{9}$?

Analysis. Since the parts of the fractions are all *ninths*, they may be added just as we add *whole* numbers; thus, 4 *ninths* and 7 *ninths* are 11 *ninths*, and 8 *ninths* are 19 *ninths*, or $\frac{19}{9}=2\frac{1}{9}$.

OPERATION.

$\frac{4}{9}+\frac{7}{9}+\frac{8}{9}=\frac{19}{9}=2\frac{1}{9}$

Ans. $2\frac{1}{9}$.

Hence we employ the following

RULE.—*Add the numerators, and place their sum over the common denominator.*

EXAMPLES.

2. Find the sum of $\frac{3}{7}$, $\frac{5}{7}$, $\frac{6}{7}$ and $\frac{4}{7}$. *Ans.* $2\frac{4}{7}$.

3. Add $\frac{8x}{10}$, $\frac{7x}{10}$, $\frac{5x}{10}$ and $\frac{3x}{10}$. *Ans.* $\frac{23x}{10}=2x+\frac{3x}{10}$.

4. Add $\frac{5a}{6}$, $\frac{4a}{6}$, $\frac{3a}{6}$ and $\frac{5a}{6}$. *Ans.* $\frac{17a}{6}=2a+\frac{5a}{6}$.

5. Add $\frac{5y}{8}$, $\frac{4y}{8}$, $\frac{7y}{8}$ and $\frac{6y}{8}$. *Ans.* $2y+\frac{6y}{8}$.

6. Add $\frac{3a}{x}$, $\frac{5a}{x}$, $\frac{7a}{x}$ and $\frac{4a}{x}$. *Ans.* $\frac{19a}{x}$.

7. Add $\frac{5x}{4b}$, $\frac{3x}{4b}$, $\frac{6x}{4b}$ and $\frac{7x}{4b}$. *Ans.* $\frac{21x}{4b}$.

8. Add $\frac{2b}{3a}$, $\frac{c}{3a}$ and $\frac{5x}{3a}$. *Ans.* $\frac{2b+c+5x}{3a}$.

9. What is the sum of $\frac{2a-x}{7c}$ and $\frac{3a+x}{7c}$? *Ans.* $\frac{5a}{7c}$.

10. Add $\frac{3x+y}{4b}$, $\frac{2x}{4b}$ and $\frac{5x-y}{4b}$. *Ans.* $\frac{10x}{4b}$.

11. Add $\frac{3ax}{5by}$, $\frac{2ad}{5by}$ and $\frac{4ax}{5by}$. *Ans.* $\frac{7ax+2ad}{5by}$.

CASE II.

137. When the Denominators are Unlike.

Ex. 1. What is the sum of $\frac{1}{2}$, $\frac{3}{4}$ and $\frac{5}{6}$?

Analysis. We cannot add these fractions in their present shape, because they do not express *like* parts of a unit; we must therefore first reduce them to a *common* denominator. We see that 12 is the *least* common multiple of the denominators. Reducing them all to *twelfths*, as previously explained, we have $\frac{1}{2}=\frac{6}{12}$, $\frac{3}{4}=\frac{9}{12}$ and $\frac{5}{6}=\frac{10}{12}$. Then $\frac{6}{12}$ and $\frac{9}{12}$ are $\frac{15}{12}$, and $\frac{10}{12}$ are $\frac{25}{12}=2\frac{1}{12}$. *Ans.* $2\frac{1}{12}$.

2. What is the sum of $\frac{a}{c}$ and $\frac{b}{x}$?

Analysis. We first reduce them to a common denominator: cx is the least common denominator. Then, $\frac{a}{c}=\frac{ax}{cx}$ and $\frac{b}{x}=\frac{bc}{cx}$. Then, adding the numerators, we have $\frac{ax+bc}{cx}$. *Ans.* $\frac{ax+bc}{cx}$.

Hence, we derive the following

RULE.—*Reduce the fractions to a common denominator, add the numerators, and place the sum over the common denominator.*

EXAMPLES.

3. Add together $\frac{3x}{4}$, $\frac{5x}{8}$ and $\frac{9x}{12}$. *Ans.* $\frac{51x}{24}=2x+\frac{3x}{24}$.

4. Add $\frac{x}{2}$, $\frac{3x-2}{4}$ and $\frac{5x+3}{6}$. *Ans.* $2x+\frac{x}{12}$.

5. Add $\frac{3a}{4b}$, $\frac{5c}{6d}$ and $\frac{7x}{8}$. *Ans.* $\frac{18ad+20bc+21bdx}{24bd}$.

6. Add $\frac{2a}{3m}$, $\frac{4b}{6n}$ and $\frac{8c}{9mn}$. *Ans.* $\frac{12an+12bm+16c}{18mn}$.

7. Add $\frac{a}{x}$, $\frac{b}{y}$ and $\frac{c}{z}$. *Ans.* $\frac{ayz+bxz+cxy}{xyz}$.

8. Find the sum of $\frac{3x}{5}$, $\frac{4x-3}{10}$ and $\frac{5x+3}{6}$.

Ans. $\frac{55x+6}{30}$.

9. Find the sum of $\frac{2x}{5}$, $\frac{3x+5}{4}$ and $\frac{9x}{10}$. *Ans.* $\frac{41x+25}{20}$

10. Add $\frac{x+m}{3}$, $\frac{5x}{6}$ and $\frac{2x-3m}{9}$. *Ans.* $\frac{25x}{18}$

11. Find the sum of $\frac{a}{x}$, $\frac{b}{y}$ and $\frac{c}{xy}$. *Ans.* $\frac{ay+bx+c}{xy}$.

12. Add together $\frac{5x}{9}$, $\frac{3x+a}{12}$ and $\frac{5x-a}{6}$. *Ans.* $\frac{59x-3a}{36}$.

13. Find the sum of $\frac{2x}{3}$, $\frac{3x}{5}$, $\frac{7x}{10}$ and $\frac{14x}{15}$. *Ans.* $2x+\frac{27x}{30}$.

14. Find the sum of $\frac{3a}{4}$, $\frac{5a}{8}$, $\frac{7a}{12}$ and $\frac{9a}{16}$. *Ans.* $2a+\frac{25a}{48}$.

15. What is the sum of $3a+\frac{4b}{5}$ and $4a+\frac{2b}{3}$?

Ans. $7a+\frac{22b}{15}$.

When there are *mixed* quantities, as in the example just given, we may add the fractional parts, and unite their sum to that of the integral parts.

16. What is the sum of $3x+\frac{2a}{3}$, $4x+\frac{2a}{7}$ and $5x$?

Ans. $12x+\frac{20a}{21}$.

17. Find the sum of $4y-\frac{3x}{4}$, $5y+\frac{2x}{3}$ and $3y$.

Ans. $12y-\frac{x}{12}$.

18. Find the sum of $2x+\frac{4b}{5}$, $3x-\frac{2b}{3}$, and $4x$.

Ans. $9x+\frac{2b}{15}$.

19. Find the sum of $3y+\frac{3a-x}{6}$, $2y+\frac{2a+x}{3}$ and $5y$.

Ans. $10y+\frac{7a+x}{6}$.

20. Find the sum of $2x+\frac{2y-a}{7}$, $3x+\frac{3y-2a}{5}$ and $4x$.

Ans. $9x+\frac{31y-19a}{35}$.

21. Find the sum of $3a+\frac{2x-y}{3}$, $2a$, and $4a+\frac{2x+3y}{9}$.

Ans. $9a+\frac{8x}{9}$.

22. Find the sum of $5x+\frac{3y-a}{5}$, $3x$ and $3x+\frac{2y+3a}{15}$.

Ans. $11x+\frac{11y}{15}$.

SUBTRACTION OF FRACTIONS.

CASE I.

138. When the Denominators are Alike.

Ex. 1. If $\frac{2}{7}$ be taken from $\frac{6}{7}$, what will remain?

Analysis. Since the parts of the fractions are all *sevenths*, they may be taken from each other just as whole numbers may be subtracted from each other; thus, 2 *sevenths* from 6 *sevenths* leave 4 *sevenths*. *Ans.* $\frac{4}{7}$.

OPERATION.

$\frac{6}{7}-\frac{2}{7}=\frac{4}{7}$

Hence, we employ the following

RULE.—*Take the less numerator from the greater, and place the difference over the common denominator.*

EXAMPLES.

2. From $\frac{8}{9}$ take $\frac{3}{9}$. *Ans.* $\frac{5}{9}$.

3. From $\frac{15}{17}$ take $\frac{8}{17}$. *Ans.* $\frac{7}{17}$.

4. From $\frac{7a}{8}$ take $\frac{2a}{8}$. *Ans.* $\frac{5a}{8}$.

5. From $\frac{7x}{10}$ take $\frac{3x}{10}$. *Ans.* $\frac{4x}{10}$.

6. From $\frac{9y}{11a}$ take $\frac{5y}{11a}$. *Ans.* $\frac{4y}{11a}$.

7. From $\frac{7ab}{9xy}$ take $\frac{3ab}{9xy}$. *Ans.* $\frac{4ab}{9xy}$.

CASE II.

139. When the Denominators are Unlike.

Ex. 1. From $\frac{6}{7}$ take $\frac{2}{3}$.

Analysis. We cannot subtract these fractions in their present shape, for 2 *thirds* taken from 6 *sevenths* will leave neither 4 *thirds* nor 4 *sevenths*, but by reducing them to a common denominator (127) we have $\frac{6}{7}=\frac{18}{21}$ and $\frac{2}{3}=\frac{14}{21}$; and now, subtracting $\frac{14}{21}$ from $\frac{18}{21}$, we have $\frac{4}{21}$ as the remainder. *Ans.* $\frac{4}{21}$.

OPERATION.

$\frac{6}{7}-\frac{2}{3}=\frac{18}{21}-\frac{14}{21}=\frac{4}{21}$

2. From $\frac{5x}{6}$ take $\frac{2x}{9}$.

Analysis. Reducing the fractions to their *least* common denominator, we have $\frac{5x}{6}=\frac{15x}{18}$ and $\frac{2x}{9}=\frac{4x}{18}$; then $\frac{4x}{18}$ from $\frac{15x}{18}$ will leave $\frac{11x}{18}$. *Ans.* $\frac{11x}{18}$.

OPERATION.

$$\frac{5x}{6}-\frac{2x}{9}=\frac{15x}{18}-\frac{4x}{18}=\frac{11x}{18}$$

From the foregoing explanations we deduce the following

RULE.—*Reduce the fractions to a common denominator; take the less numerator from the greater, and place the difference over the common denominator.*

EXAMPLES.

3. From $\frac{9x}{10}$ subtract $\frac{3x}{4}$. *Ans.* $\frac{3x}{20}$.

4. From $\frac{x}{2}$ subtract $\frac{2x}{5}$. *Ans.* $\frac{x}{10}$.

5. From $\frac{3x+5}{4}$ take $\frac{3x}{8}$. *Ans.* $\frac{3x+10}{8}$.

6. From $\frac{3x-4}{5}$ take $\frac{x}{3}$. *Ans.* $\frac{4x-12}{15}$.

7. From $\frac{a}{b}$ take $\frac{c}{d}$. *Ans.* $\frac{ad-bc}{bd}$.

8. From $\frac{ax}{by}$ take $\frac{m}{p}$. *Ans.* $\frac{apx-bmy}{bpy}$.

9. From $\frac{3x+5}{3}$ take $\frac{3x-4}{6}$. *Ans.* $\frac{3x+14}{6}$.

10. From $\frac{7x+5}{8}$ take $\frac{5x+4}{12}$. *Ans.* $\frac{11x+7}{24}$.

11. From $\frac{7x-4}{10}$ take $\frac{3x+2}{6}$. *Ans.* $\frac{6x-22}{30}$.

12. From $\frac{5x+4}{6}$ take $\frac{5x+4}{9}$. *Ans.* $\frac{5x+4}{18}$.

13. From $\frac{x+1}{2}$ take $\frac{x-2}{3}$. *Ans.* $\frac{x+7}{6}$.

14. From $\frac{x-2}{3}$ take $\frac{x+3}{4}$. *Ans.* $\frac{x-17}{12}$.

15. From $\left(3x-\frac{3a+4}{5}\right)$ take $\left(2x+\frac{2a-3}{4}\right)$.

OPERATION.

$$\left(3x-\frac{3a+4}{5}\right)-\left(2x+\frac{2a-3}{4}\right)$$
$$\left(\frac{15x-3a-4}{5}\right)-\left(\frac{8x+2a-3}{4}\right)$$
$$\left(\frac{60x-12a-16}{20}\right)-\left(\frac{40x+10a-15}{20}\right)$$
$$\frac{20x-22a-1}{20}\ \textit{Ans.}$$

Analysis. Reducing the *mixed* quantities to a fractional form, we have $3x-\frac{3a+4}{5}=\frac{15x-3a-4}{5}$ and $2x+\frac{2a-3}{4}=\frac{8x+2a-3}{4}$. Then reducing them to a common denominator and subtracting the numerators, we have $\frac{20x-22a-1}{20}$. *Ans.* $\frac{20x-22a-1}{20}$.

16. From $\left(5x-\frac{3a+2}{3}\right)$ take $\left(2x+\frac{2a-1}{4}\right)$.

Ans. $\frac{36x-18a-5}{12}$.

17. From $\left(3x+\frac{2n-3}{5}\right)$ take $\left(2x-\frac{3n+5}{3}\right)$.

Ans. $\frac{15x+21n+16}{15}$

18. From $\left(5y-\frac{2b+3}{3}\right)$ take $\left(2y-\frac{3b-2}{4}\right)$.

Ans. $\frac{36y+b-18}{12}$.

19. From $\left(4x+\frac{5c-4}{6}\right)$ take $\left(2x+\frac{4c+5}{9}\right)$.

Ans. $\frac{36x+7c-22}{18}$.

20. From $\left(5a+\frac{2x-3}{3}\right)$ take $\left(4a-\frac{3x+2}{2}\right)$.

Ans. $\frac{6a+13x}{6}$.

21. From $\left(3y+\frac{3x-5}{5}\right)$ take $\left(2y+\frac{2x-4}{4}\right)$. *Ans.* $\frac{10y+x}{10}$.

MULTIPLICATION OF FRACTIONS.

140. To Multiply a Fraction by an Integer or Entire Quantity.

Ex. 1. What would 3 bushels of oats cost, at $\frac{5}{6}$ of a dollar per bushel?

Analysis. If 1 bushel cost $\frac{5}{6}$ of a dollar, then 3 bushels would cost 3 times as much $=\frac{15}{6}$ of a dollar $=2\frac{3}{6}$ dollars $=2\frac{1}{2}$ dollars. Here we *multiplied* the *numerator* of the fraction by 3, but if we had *divided* the *denominator* we should have obtained $\frac{5}{2}$ of a dollar $=2\frac{1}{2}$ dollars, the same as before. *Ans.* $2\frac{1}{2}$ dolls.

OPERATION.

$\frac{5}{6}\times3=\frac{15}{6}=2\frac{3}{6}=2\frac{1}{2}$ dolls.

Or,

$\frac{5}{6}\times3=\frac{5}{2}=2\frac{1}{2}$ dolls.

2. Multiply $\frac{7}{10}$ by 5.

Analysis. 5 times $\frac{1}{10}=\frac{5}{10}$; then 5 times $\frac{7}{10}$ is equal to 7 *times* as much $=\frac{35}{10}=3\frac{5}{10}=3\frac{1}{2}$. Here we *multiplied* the *numerator* of the fraction by 5, but if we had *divided* the *denominator* we should have obtained $\frac{7}{2}=3\frac{1}{2}$, the same as before. *Ans.* $3\frac{1}{2}$.

OPERATION.

$\frac{7}{10}\times5=\frac{35}{10}=3\frac{5}{10}=3\frac{1}{2}$

Or,

$\frac{7}{10}\times5=\frac{7}{2}=3\frac{1}{2}$

3. Multiply $\frac{ax}{bc}$ by c.

Analysis. Here we *multiplied* the *numerator* by c, and obtained for the product $\frac{ax}{b}$; but had we *divided* the *denominator*, we should have obtained $\frac{ax}{b}$, the same result as before. *Ans.* $\frac{ax}{b}$.

OPERATION.

$\frac{ax}{bc}\times c=\frac{acx}{bc}=\frac{ax}{b}$

Or,

$\frac{ax}{bc}\times c=\frac{ax}{b}$

It will be observed that the resulting product is the same whether we multiply the numerator or divide the denominator.

From these illustrations we derive the following

RULE.—*Multiply the numerator by the integral quantity, or divide the denominator, when it can be done without a remainder.*

EXAMPLES.

4. Multiply $\frac{c}{d}$ by x. *Ans.* $\frac{cx}{d}$.

5. Multiply $\frac{4c}{15x}$ by 3. *Ans.* $\frac{4c}{5x}$.

6. Multiply $\frac{4x}{5y}$ by $3y$. *Ans.* $\frac{12x}{5}$.

7. Multiply $\frac{7ab}{12xy}$ by $4x$. *Ans.* $\frac{7ab}{3y}$.

8. Multiply $\frac{5ax}{7b}$ by $4y$. *Ans.* $\frac{20axy}{7b}$.

9. Multiply $\frac{5xy}{12a}$ by $3ac$. *Ans.* $\frac{5cxy}{4}$.

10. Multiply $\frac{a+b}{a-b}$ by $5x$. *Ans.* $\frac{5ax+5bx}{a-b}$.

11. Multiply $\frac{x-5}{8by}$ by $4b$. *Ans.* $\frac{x-5}{2y}$.

12. Multiply $\frac{2x-3+5c}{3ab}$ by $5n$. *Ans.* $\frac{10nx-15n+25cn}{3ab}$.

13. Multiply $\frac{5x-4a+3}{a^2-x^2}$ by $(a-x)$. *Ans.* $\frac{5x-4a+3}{a+x}$.

14. Multiply $\frac{3a+2b-7}{n^2-m^2}$ by $(n+m)$. *Ans.* $\frac{3a+2b-7}{n-m}$.

15. Multiply $\frac{b+x}{3ab-2y}$ by $(b-x)$. *Ans.* $\frac{b^2-x^2}{3ab-2y}$.

16. Multiply $\frac{x+5}{y-3}$ by $(x-3)$. *Ans.* $\frac{x^2+2x-15}{y-3}$.

17. Multiply $\frac{x-4}{y+5}$ by $(x+6)$. *Ans.* $\frac{x^2+2x-24}{y+5}$.

18. Multiply $\frac{7x}{9}$ by 9. *Ans.* $\frac{63x}{9}=7x$.

19. Multiply $\frac{5ab}{7y}$ by $7y$. *Ans.* $\frac{35aby}{7y}=5ab$.

20. Multiply $\frac{ax}{3b}$ by $3b$. *Ans.* $\frac{3abx}{3b}=ax$.

From the last three examples we derive a principle of considerable practical value—viz.:

141. *A fraction is multiplied by a quantity equal to its denominator simply by removing, or cancelling, the denominator.*

CASE II.

142. To Multiply an Integer, or Entire Quantity, by a Fraction.

Ex. 1. If a yard of muslin cost 32 cents, what would be the cost of $\frac{5}{8}$ of a yard?

Analysis. If 1 yard cost 32 cents, then $\frac{1}{8}$ of a yard would cost *one-eighth* of 32 cents = 4 cents; and $\frac{5}{8}$ of a yard would cost 5 times as much as $\frac{1}{8}$, which would be 20 cents.

Or we might say: Since one yard costs 32 cents, $\frac{5}{8}$ of a yard will cost $\frac{5}{8}$ times as much, which is $\frac{160}{8}$ cents = 20 cents.

OPERATION.

yd.

$1 = 32$ cents.

$\frac{1}{8} = 4$ "

$\frac{5}{8} = 20$ "

Or,

$32 \times \frac{5}{8} = \frac{160}{8} = 20$ cents.

It will readily be seen that here we have divided the whole number by the denominator, 8, and then multiplied by the numerator 5; but the result will be the same if we multiply first and then divide; and, in practice, this is the usual method.

From this illustration we derive the following

RULE.—*Multiply the integral quantity by the numerator, and divide by the denominator.*

EXAMPLES.

2. Multiply 45 by $\frac{5}{6}$. *Ans.* $37\frac{1}{2}$.

3. Multiply $36x$ by $\frac{5}{9}$. *Ans.* $20x$.

4. Multiply $49ax^2$ by $\frac{6}{7}$. *Ans.* $42ax^2$.

5 Multiply $27ab$ by $\frac{3x}{5y}$. *Ans.* $\frac{81abx}{5y}$.

6. Multiply $(3a+5x)$ by $\frac{7a}{10x}$. *Ans.* $\frac{21a^2+35ax}{10x}$.

7. Multiply $(4x-2a+5)$ by $\frac{5a}{7xy}$. *Ans.* $\frac{20ax-10a^2+25a}{7xy}$.

8. Multiply (m^2-n^2) by $\frac{4a}{m-n}$. *Ans.* $4a(m+n)$.

9. Multiply (a^2-b^2) by $\frac{3x}{a+b}$. *Ans.* $3x(a-b)$.

10. Multiply (x^2-a^2) by $\frac{5c}{x-a}$. *Ans.* $5c(x+a)$.

11. Multiply $(1-x^2)$ by $\frac{9b}{1+x}$. *Ans.* $9b(1-x)$.

12. Multiply $(3a^3+5a^2-7a)$ by $\frac{7x}{9a}$.

Ans. $\frac{21a^2x+35ax-49x}{9}$.

13. Multiply $(5x^3-7x^2+9x)$ by $\frac{5b}{7x}$.

Ans. $\frac{25bx-35bx^2+45b}{7}$.

CASE III.

143. To Multiply a Fraction by a Fraction.

Ex. 1. Multiply $\frac{2}{3}$ by $\frac{4}{5}$.

Analysis. $\frac{1}{3}$ of $\frac{1}{5}$ (133) $=\frac{1}{15}$; then $\frac{2}{3}$ of $\frac{1}{5}$ will be *twice* as much $=\frac{2}{15}$; and $\frac{2}{3}$ of $\frac{4}{5}$ will be *four* times as much $=\frac{8}{15}$.

OPERATION.

$\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$ *Ans.*

From the Analysis we deduce the following

RULE.—*Multiply the numerators together for a new numerator, and the denominators together for a new denominator.*

NOTE.—Mixed quantities should be reduced to fractional forms before multiplying.

EXAMPLES.

2. Multiply $\frac{4x}{5y}$ by $\frac{3a}{7b}$. *Ans.* $\frac{12ax}{35by}$.

3. Multiply $\frac{3a}{4c}$ by $\frac{4x}{5m}$. *Ans.* $\frac{3ax}{5cm}$.

4. Multiply $\frac{5x}{7a}$ by $\frac{7c}{9y}$. *Ans.* $\frac{5cx}{9ay}$.

5. Multiply $\frac{7a}{8b}$ by $\frac{5m}{7c}$. *Ans.* $\frac{5am}{8bc}$.

6. Multiply $\frac{3ax}{5b}$ by $\frac{5m}{7ay}$. *Ans.* $\frac{3mx}{7by}$.

7. What is the value of $\frac{3a}{4b}\times\frac{4x}{5y}\times\frac{5b}{7c}$? *Ans.* $\frac{3ax}{7cy}$.

8. What is the value of $\frac{3x}{5}\times\frac{5a}{6}\times\frac{4b}{7a}$? *Ans.* $\frac{2bx}{7}$.

9. What is the value of $\frac{4x}{7}\times\frac{14x}{15a}\times\frac{3a}{8x}$? *Ans.* $\frac{x}{5}$.

10. Multiply $\frac{3x-7}{4y}$ by $\frac{4a}{9c}$. $\frac{3ax-7a}{9cy}$.

11. Multiply $\frac{5x+9}{3x-2}$ by $\frac{5a}{7b}$. *Ans.* $\frac{25ax+45a}{21bx-14b}$.

12. Multiply $\frac{4x-5}{7a}$ by $\frac{3b}{3x+4}$. *Ans.* $\frac{12bx-15b}{21ax+28a}$.

13. Multiply $\frac{5ax}{4y-2}$ by $\frac{4b+1}{6b}$. *Ans.* $\frac{20abx+5ax}{24by-12b}$.

14. Multiply $\frac{3x-5}{4y-2}$ by $\frac{2x+1}{3y-3}$. *Ans.* $\frac{6x^2-7x-5}{12y^2-6y-6}$.

15. Multiply $2x+\frac{3a}{4}$ by $\frac{4x}{7y}$. *Ans.* $\frac{8x^2+3ax}{7y}$.

16. Multiply $3y-\frac{5b}{7}$ by $\frac{7a}{9x}$. *Ans.* $\frac{21ay-5ab}{9x}$.

17. Multiply $3a+\frac{3x-4}{5}$ by $\frac{5a}{6b}$. *Ans.* $\frac{15a^2+3ax-4a}{6b}$.

18. Multiply $5y-\frac{2a-3}{7}$ by $\frac{7x}{10c}$. *Ans.* $\frac{35xy-2ax+3x}{10c}$.

DIVISION OF FRACTIONS.

CASE I.

144. To Divide a Fraction by an Integer.

Ex. 1. If 7 yards of cloth cost $\frac{14}{15}$ of a dollar, what would be the cost of 1 yard?

Analysis. If 7 yards cost $\frac{14}{15}$ of a dollar, 1 yard would cost $\frac{1}{7}$ as much, which would be $\frac{2}{15}$ of a dollar.

Ans. $\frac{2}{15}$ of a dollar.

OPERATION.

7 yd. $= \frac{14}{15}$

1 yd. $= \frac{2}{15}$

In this example we *divided* the *numerator*, 14, by 7, and obtained $\frac{2}{15}$ as the result. If we had *multiplied* the *denominator*, 15, by 7, we should have had $\frac{14}{105}$, which, reduced to its lowest terms, gives $\frac{2}{15}$, the same as before.

2. Divide $\frac{6ax}{7b}$ by $3a$.

Analysis. Here we *divided* the *numerator*, $6ax$, by $3a$, and obtained $\frac{2x}{7b}$ as the quotient. Had we *multiplied* the denominator, $7b$, by $3a$, we should have had $\frac{6ax}{21ab}$, which, reduced to its lowest terms, gives $\frac{2x}{7b}$, the same as before.

OPERATION.

$$\frac{6ax}{7b} \div 3a = \frac{2x}{7b}$$

Ans. $\frac{2x}{7b}$.

From the foregoing explanations we derive the following

RULE.—*Divide the numerator or multiply the denominator by the integer.*

EXAMPLES.

3. Divide $\frac{15}{17}$ by 3. *Ans.* $\frac{5}{17}$.
4. Divide $\frac{2}{3}$ by 4. *Ans.* $\frac{1}{6}$.
5. Divide $\frac{15cd}{19xy}$ by $5c$. *Ans.* $\frac{3d}{19xy}$.
6. Divide $\frac{3ax}{4b}$ by $7n$. *Ans.* $\frac{3ax}{28bn}$.
7. Divide $\frac{10cd}{3ax}$ by $5d$. *Ans.* $\frac{2c}{3ax}$.
8. Divide $\frac{5xy}{6ab}$ by $3m$. *Ans.* $\frac{5xy}{18abm}$.
9. Divide $\frac{5x^2-5x}{8ac}$ by $5x$. *Ans.* $\frac{x-1}{8ac}$.
10. Divide $\frac{3x-2}{4x+3}$ by $2a$. *Ans.* $\frac{3x-2}{8ax+6a}$.
11. Divide $\frac{6y^2+9y}{3a+2b}$ by $3y$. *Ans.* $\frac{2y+3}{3a+2b}$.
12. Divide $\frac{4a^2-3a+5}{5a-1}$ by $4a$. *Ans.* $\frac{4a^2-3a+5}{20a^2-4a}$.
13. Divide $\frac{6a^3-9a^2+3a}{4ab-2x}$ by $3a$. *Ans.* $\frac{2a^2-3a+1}{4ab-2x}$.
14. Divide $5x+\frac{10x}{3b}$ by $5x$. *Ans.* $\frac{3b+2}{3b}$.
15. Divide $4y-\frac{8y}{5a}$ by $4y$. *Ans.* $\frac{5a-2}{5a}$.
16. Divide $3a+\frac{2x-5}{3x}$ by $5x$. *Ans.* $\frac{9ax+2x-5}{15x^2}$.

17. Divide $2y^2-\frac{3y+9y^3}{6}$ by $3y$. *Ans.* $\frac{4y-1-3y^2}{6}$.

18. Divide $3x^2-\frac{6x-12x^3}{4}$ by $6x$. *Ans.* $\frac{2x^2-1+2x}{4}$.

CASE II.

145. To Divide an Integer by a Fraction.

Ex. 1. Divide 7 by $\frac{3}{4}$.

Analysis. 1 unit is contained in 7 units 7 times; then $\frac{1}{4}$ of a unit will be contained in 7 units 4 times as often, which will be 28 times; and $\frac{3}{4}$ will be contained only *one third* as many times, which will be $\frac{28}{3}$ times $=9\frac{1}{3}$ times. *Ans.* $9\frac{1}{3}$.

OPERATION.

$7\div 1=7$

$7\div\frac{1}{4}=7\times 4=28$

$7\div\frac{3}{4}=\frac{1}{3}$ of $28=\frac{28}{3}=9\frac{1}{3}$

2. Divide x by $\frac{4}{5}$.

Analysis. 1 unit is contained in x, x times; then $\frac{1}{5}$ of a unit will be contained 5 *times* as often $=5x$ times; and $\frac{4}{5}$ will be contained only *one-fourth* as many times $=\frac{5x}{4}$ times. *Ans.* $\frac{5x}{4}$.

OPERATION.

$x\div 1=x$

$x\div\frac{1}{5}=5\times x=5x$

$x\div\frac{4}{5}=\frac{1}{4}$ of $5x=\frac{5x}{4}$

3. Divide y by $\frac{m}{n}$.

Analysis. 1 unit is contained in y, y times; then $\frac{1}{n}$ of a unit will be contained n *times* as often, which will be ny times; and $\frac{m}{n}$ will be contained only $\frac{1}{m}$th as many times $=\frac{ny}{m}$ times. *Ans.* $\frac{ny}{m}$.

OPERATION.

$y\div 1=y$

$y\div\frac{1}{n}=n\times y=ny$

$y\div\frac{m}{n}=\frac{1}{m}$ of $ny=\frac{ny}{m}$

By examining the Analysis we see that the operation consists in multiplying the integral quantity by the denominator of the fraction, and dividing the product by the numerator. Hence, we have the following

RULE.—*Multiply the integer by the denominator of the fraction, and divide the product by the numerator.*

EXAMPLES.

4. Divide 21 by $\frac{5}{6}$. *Ans.* $25\frac{1}{5}$.

5. Divide $3a$ by $\frac{6y}{7x}$. *Ans.* $\frac{7ax}{2y}$.

6. Divide $6x^2$ by $\frac{3ax}{4n}$. *Ans.* $\frac{8nx}{a}$.

7. Divide $2a^2$ by $\frac{4ab}{7xy}$. *Ans.* $\frac{7axy}{2b}$.

8. Divide $9y^2$ by $\frac{6b}{11an}$. *Ans.* $\frac{33any^2}{2b}$.

9. Divide $5ax$ by $\frac{b-x}{4n}$. *Ans.* $\frac{20anx}{b-x}$.

10. Divide $3n$ by $\frac{c-d}{a-x}$. *Ans.* $\frac{3an-3nx}{c-d}$.

11. Divide 15 by $2\frac{2}{3}$. *Ans.* $5\frac{5}{8}$.

12. Divide $5x$ by $\left(a+\frac{x}{3}\right)$. *Ans.* $\frac{15x}{3a+x}$.

13. Divide $3b$ by $\left(x-\frac{b}{4}\right)$. *Ans.* $\frac{12b}{4x-b}$.

14. Divide $7a$ by $\left(3x+\frac{2a-3}{5}\right)$. *Ans.* $\frac{35a}{15x+2a-3}$.

15. Divide $10ay$ by $\left(5a-\frac{3x+5}{7}\right)$. *Ans.* $\frac{70ay}{35a-3x-5}$.

16. Divide $9ax$ by $\left(3y+\frac{4a-7}{3}\right)$. *Ans.* $\frac{27ax}{9y+4a-7}$.

17. Divide $13nx$ by $\left(6a-\frac{3x-5}{5}\right)$. *Ans.* $\frac{65nx}{30a-3x+5}$.

CASE III.

146. To Divide a Fraction by a Fraction.

Ex. 1. Divide $\frac{5}{7}$ by $\frac{3}{4}$.

Analysis. 1 unit is contained in $\frac{5}{7}$, *five-sevenths* of a time; then $\frac{1}{4}$ of a unit will be contained 4 *times* as often $=\frac{20}{7}$ times; and $\frac{3}{4}$ will be contained only *one-third* as many times, which will be $\frac{20}{21}$ times. *Ans.* $\frac{20}{21}$.

OPERATION.

$$\frac{5}{7}\div 1=\frac{5}{7}$$
$$\frac{5}{7}\div\frac{1}{4}=4\times\frac{5}{7}=\frac{20}{7}$$
$$\frac{5}{7}\div\frac{3}{4}=\frac{1}{3}\text{ of }\frac{20}{7}=\frac{20}{21}$$

2. Divide $\frac{a}{b}$ by $\frac{4}{5}$.

Analysis. 1 unit is contained in $\frac{a}{b}$, $\frac{a}{b}$ times; then $\frac{1}{5}$ of a unit will be contained 5 times as often $=\frac{5a}{b}$ times; and $\frac{4}{5}$ will be contained only *one-fourth* as many times $=\frac{5a}{4b}$ times. *Ans.* $\frac{5a}{4b}$.

OPERATION.

$$\frac{a}{b}\div 1=\frac{a}{b}$$
$$\frac{a}{b}\div\frac{1}{5}=5\times\frac{a}{b}=\frac{5a}{b}$$
$$\frac{a}{b}\div\frac{4}{5}=\frac{1}{4}\text{ of }\frac{5a}{b}=\frac{5a}{4b}$$

3. Divide $\frac{c}{d}$ by $\frac{m}{n}$.

Analysis. 1 unit is contained in $\frac{c}{d}$, $\frac{c}{d}$ times; then $\frac{1}{n}$th of a unit will be contained *n times* as often $=\frac{cn}{d}$ times; and $\frac{m}{n}$ will be contained only $\frac{1}{m}$th as many times $=\frac{cn}{dm}$ times. *Ans.* $\frac{cn}{dm}$.

OPERATION.

$$\frac{c}{d}\div 1=\frac{c}{d}$$
$$\frac{c}{d}\div\frac{1}{n}=n\times\frac{c}{d}=\frac{cn}{d}$$
$$\frac{c}{d}\div\frac{m}{n}=\frac{1}{m}\text{ of }\frac{cn}{d}=\frac{cn}{dm}$$

In all these examples the operation consists in multiplying the dividend by the divisor inverted. Hence, the

RULE.—*Invert the divisor, and multiply the numerators together for a numerator, and the denominators together for a denominator.*

Note.—Mixed quantities must be reduced to fractional forms, and compound fractions to simple ones, before inverting or multiplying.

EXAMPLES.

4. Divide $\frac{8}{15}$ by $\frac{4}{5}$. *Ans.* $\frac{2}{3}$.

5. Divide $\frac{3a}{4x}$ by $\frac{4b}{5c}$. *Ans.* $\frac{15ac}{16bx}$.

6. Divide $\frac{4b}{5y}$ by $\frac{3x}{4a}$. *Ans.* $\frac{16ab}{15xy}$.

7. Divide $\frac{2x}{3a-4}$ by $\frac{5x}{7y}$. *Ans.* $\frac{14y}{15a-20}$.

8. Divide $\frac{3b}{4x-5}$ by $\frac{4y}{5x}$. *Ans.* $\frac{15bx}{16xy-20y}$.

9. Divide $\frac{3x-7}{4a+3}$ by $\frac{5b}{6c}$. *Ans.* $\frac{18cx-42c}{20ab+15b}$.

10. Divide $\left(a+\frac{2x}{3}\right)$ by $\frac{4b}{5c}$. *Ans.* $\frac{15ac+10cx}{12b}$.

Note.—Reduce the *mixed* quantity to a fractional form.

11. Divide $\left(3b-\frac{4x}{5}\right)$ by $\frac{3c}{4n}$. $\frac{60bn-16nx}{15c}$.

12. Divide $\left(4a+\frac{2x-5}{3}\right)$ by $\frac{5n}{6p}$. *Ans.* $\frac{72ap+12px-30p}{15n}$.

13. Divide $\left(3y-\frac{4a-3}{5}\right)$ by $\frac{4x}{7b}$.

Ans. $\frac{105by-28ab+21b}{20x}$.

14. Divide $\left(5b-\frac{3x+7}{4}\right)$ by $\frac{3n}{5a}$.

Ans. $\frac{100ab-15ax-35a}{12n}$.

15. Divide $\left(5x-\frac{4a-1}{3}\right)$ by $\frac{6c}{7m}$.

Ans. $\frac{105mx-28am+7m}{18c}$.

16. Divide $\left(3x-\frac{4c-5}{6}\right)$ by $\frac{x+c-3}{4}$.

Ans. $\frac{72x-16c+20}{6x+6c-18}$.

17. Divide $\left(4a-\frac{2x+3}{7}\right)$ by $\left(5x+\frac{3a-5}{4}\right)$.

Ans. $\frac{112a-8x-12}{140x+21a-35}$.

CASE IV.

147. To Reduce COMPLEX to Simple Fractions.

NOTE.—Complex fractions only present the subject of division of fractions under all its different forms, and therefore admit of the same analyses.

Ex. 1. Reduce $\frac{\frac{3}{4}}{\frac{5}{7}}$ to a simple fraction.

Analysis. The complex fraction $\frac{\frac{3}{4}}{\frac{5}{7}}$ is the same as $\frac{3}{4}+\frac{5}{7}$, and this, according to **146**, is the same as $\frac{3}{4}\times\frac{7}{5}$, which equals $\frac{21}{20}$. *Ans.* $\frac{21}{20}$.

OPERATION.

$$\frac{\frac{3}{4}}{\frac{5}{7}}=\frac{3}{4}+\frac{5}{7}=\frac{3}{4}\times\frac{7}{5}=\frac{21}{20}$$

2. Reduce $\frac{\frac{a}{b}}{\frac{m}{n}}$ to a simple fraction.

Analysis. The complex fraction $\frac{\frac{a}{b}}{\frac{m}{n}}$ is the same as $\frac{a}{b}+\frac{m}{n}$; and this, according to **146**, is the same as $\frac{a}{b}\times\frac{n}{m}$, which is equal to $\frac{an}{bm}$. *Ans.* $\frac{an}{bm}$.

OPERATION.

$$\frac{\frac{a}{b}}{\frac{m}{n}}=\frac{a}{b}+\frac{m}{n}=\frac{a}{b}\times\frac{n}{m}=\frac{an}{bm}$$

3. Reduce $\frac{p}{\frac{a}{c}}$ to a simple fraction.

OPERATION.

$$\frac{p}{\frac{a}{c}} = p \div \frac{a}{c} = \frac{p}{1} \div \frac{a}{c} = \frac{p}{1} \times \frac{c}{a} = \frac{cp}{a}$$

Analysis. In this example we change the *numerator*, p, into a fraction by writing 1 under it, thus $\frac{p}{1}$; our complex fraction then becomes $\frac{\frac{p}{1}}{\frac{a}{c}}$; and this fraction is the same as $\frac{p}{1} \div \frac{a}{c}$, or the same as $\frac{p}{1} \times \frac{c}{a} = \frac{cp}{a}$.

Ans. $\frac{cp}{a}$.

4. Reduce $\frac{\frac{x}{a}}{c}$ to a simple fraction.

OPERATION.

$$\frac{\frac{x}{a}}{c} = \frac{x}{a} \div c = \frac{x}{a} \div \frac{c}{1} = \frac{x}{a} \times \frac{1}{c} = \frac{x}{ac}$$

Analysis. We first change the denominator, c, into a fractional form by writing 1 under it, thus $\frac{c}{1}$; our fraction now becomes $\frac{\frac{x}{a}}{\frac{c}{1}}$; and this fraction is the same as $\frac{x}{a} \div \frac{c}{1}$, or the same as $\frac{x}{a} \times \frac{1}{c} = \frac{x}{ac}$.

Ans. $\frac{x}{ac}$.

5. Reduce $\frac{a+\frac{2x}{3}}{y-\frac{4b}{5}}$ to a simple fraction.

OPERATION.

$$\frac{a+\frac{2x}{3}}{y-\frac{4b}{5}} = \frac{\frac{3a+2x}{3}}{\frac{5y-4b}{5}} = \frac{3a+2x}{3} \div \frac{5y-4b}{5} = \frac{3a+2x}{3} \times \frac{5}{5y-4b} = \frac{15a+10x}{15y-12b}$$

Analysis. We first reduced the *mixed* quantities to fractional forms, and then proceeded exactly as in the foregoing examples.

Ans. $\frac{15a+10x}{15y-12b}$.

The four *literal* examples above given embrace, in substance, all the cases that usually occur in complex fractions. By a little inspection it will be seen that their reduction consists in multiplying the *upper* and the *lower* quantities together for a numerator, and the two *middle* quantities together for a denominator.

Hence, we have the following

RULE.—*Multiply the numerator of the upper fraction by the denominator of the lower for a numerator, and the denominator of the upper fraction by the numerator of the lower for a denominator.*

NOTE.—Integral quantities and mixed quantities must first be reduced to fractional forms before applying the rule.

EXAMPLES.

6. Reduce $\frac{\frac{5}{6}}{\frac{7}{8}}$ to a simple fraction. *Ans.* $\frac{20}{21}$.

7. Reduce $\frac{\frac{2x}{5y}}{\frac{3a}{7b}}$ to a simple fraction. *Ans.* $\frac{14bx}{15ay}$.

8. Reduce $\frac{\frac{5b}{9a}}{7x}$ to a simple fraction. *Ans.* $\frac{5b}{63ax}$.

9. Reduce $\frac{3a}{\frac{4x}{5y}}$ to a simple fraction. *Ans.* $\frac{15ay}{4x}$.

10. Reduce $\frac{x-\frac{2a}{5}}{a+\frac{3y}{4}}$ to a simple fraction. *Ans.* $\frac{20x-8a}{20a+15y}$.

11. Reduce $\frac{2x-\frac{3}{5}}{4y-\frac{5}{6}}$ to a simple fraction. *Ans.* $\frac{60x-18}{120y-25}$.

12. Reduce $\dfrac{3a+\dfrac{2b-3}{4}}{4x-\dfrac{2y+5}{3}}$ to a simple fraction.

Ans. $\dfrac{36a+6b-9}{48x-8y-20}$.

13. Reduce $\dfrac{\dfrac{x+1}{a-b}}{\dfrac{a+b}{x-1}}$ to a simple fraction. *Ans.* $\dfrac{x^2-1}{a^2-b^2}$.

PROMISCUOUS EXAMPLES.

1. Reduce $\dfrac{45a^2bx}{75aby}$ to its lowest terms. *Ans.* $\dfrac{3ax}{5y}$.

2. Reduce $\dfrac{64bxy}{80b^2y}$ to its lowest terms. *Ans.* $\dfrac{4x}{5b}$.

3. Reduce $\dfrac{ax^2+3x}{5x}$ to its lowest terms. *Ans.* $\dfrac{ax+3}{5}$.

4. Reduce $\dfrac{3x^2-9x+6x^3}{3x}$ to an integral quantity.

Ans. $x-3+2x^2$.

5. Reduce $\dfrac{5x+3y-2}{5}$ to a mixed quantity.

Ans. $x+\dfrac{3y-2}{5}$.

6. Reduce $\dfrac{12x^2+4a-b}{6x}$ to a mixed quantity.

Ans. $2x+\dfrac{4a-b}{6x}$.

7. Reduce $\dfrac{15a^2-20a^3+10a}{5a}$ to an integral quantity.

Ans. $3a-4a^2+2$.

8. Reduce $4a+\dfrac{2x-5}{7}$ to a fractional form.

Ans. $\dfrac{28a+2x-5}{7}$.

9. Reduce $3x+\frac{3x-2}{5}$ to a fractional form.

Ans. $\frac{18x-2}{5}$.

10. Reduce $2x-\frac{4x+5}{8}$ to a fractional form.

Ans. $\frac{12x-5}{8}$.

11. Reduce $3y-\frac{2y^2-4}{6y}$ to a fractional form.

Ans. $\frac{16y^2+4}{6y}$.

12. Reduce 7 to a fraction whose denominator shall be 9.

Ans. $\frac{63}{9}$.

13. Reduce x to a fraction whose denominator shall be 5.

Analysis. 1 unit is equal to $\frac{5}{5}$; then x units will be x *times* as much, which will be $\frac{5x}{5}$. *Ans.* $\frac{5x}{5}$.

OPERATION.

$1=\frac{5}{5}$

$x=\frac{5x}{5}$

14. Reduce y to a fraction whose denominator shall be 7.

Ans. $\frac{7y}{7}$.

15. Reduce b to a fraction whose denominator shall be n.

Ans. $\frac{bn}{n}$.

16. If $\frac{4}{5}$ of a number is 36, what is the number?

Analysis. If 4 *fifths* of a number is 36, *one fifth* of the number is *one fourth* of 36, which is 9; and if $\frac{1}{5}$ of the number is 9, then $\frac{5}{5}$, or the *whole* of the number, will be 5 times as much, which will be 45. *Ans.* 45.

OPERATION.

$\frac{4}{5}=36$

$\frac{1}{5}=9$

$\frac{5}{5}=45$

17. If $\frac{6}{7}$ of a number is 25, what is the number?

Ans. $29\frac{1}{6}$.

18. If $\frac{5}{8}$ of a number is x, what is the number?

Ans. $\frac{8x}{5}$.

19. If $\frac{7}{9}$ of a number is b, what is the number? *Ans.* $\frac{9b}{7}$.

20. If the $\frac{m}{n}$ part of a number is x, what is the number? *Ans.* $\frac{nx}{m}$.

21. If the $\frac{r}{n}$ part of a number is y, what is the number? *Ans.* $\frac{ny}{r}$.

22. Reduce $\frac{3}{4}$ of $\frac{4}{5}$ to a simple fraction. *Ans.* $\frac{3}{5}$.

23. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{7}{8}$ to a simple fraction. *Ans.* $\frac{7}{20}$.

24. Reduce $\frac{3a}{4b}$ of $\frac{7b}{9y}$ to a simple fraction. *Ans.* $\frac{7a}{12y}$.

25. Add together $\frac{3x}{4}$, $\frac{2x}{3}$ and $\frac{7x}{12}$. *Ans.* $2x$.

26. Add together $3x+\frac{2x}{5}$ and $2x-\frac{5x}{6}$. *Ans.* $5x-\frac{13x}{30}$.

27. From $\frac{5x}{7}$ subtract $\frac{2x}{3}$. *Ans.* $\frac{x}{21}$.

28. From $\frac{3x-1}{2}$ subtract $\frac{3x+5}{5}$. *Ans.* $\frac{9x-15}{10}$.

29. From $\left(3x+\frac{2a-5}{5}\right)$ take $\left(2x-\frac{3a+4}{3}\right)$.

Ans. $\frac{15x+21a+5}{15}$.

30. From $\left(5a-\frac{3x+2}{4}\right)$ take $\left(3a+\frac{2x-3}{5}\right)$.

Ans. $\frac{40a-23x+2}{20}$.

31. Multiply $\frac{5ab}{7xy}$ by $3a$. *Ans.* $\frac{15a^2b}{7xy}$.

32. Multiply $\frac{3x-7}{8ab}$ by $4b$. *Ans.* $\frac{3x-7}{2a}$.

33. Multiply $\frac{3a}{7y}$ by $\frac{5x}{6c}$. *Ans.* $\frac{5ax}{14cy}$.

34. Multiply $\frac{3x-5}{7a}$ by $\frac{6b}{9y+5}$. *Ans.* $\frac{18bx-30b}{63ay+35a}$.

35. Multiply $\left(2x-\frac{4a+3}{5}\right)$ by $\frac{4b}{7n}$.

Ans. $\frac{40bx-16ab-12b}{35n}$.

36. Divide $\frac{6ab}{7x}$ by $3b$. *Ans.* $\frac{2a}{7x}$.

37. Divide $\frac{5xy}{6ab}$ by $3a$. *Ans.* $\frac{5xy}{18a^2b}$.

38. Divide $\frac{3x-2}{5a+b}$ by $\frac{3y}{7c}$. *Ans.* $\frac{21cx-14c}{15ay+3by}$.

39. Divide $\frac{5x-4}{6c}$ by $\frac{5b-7}{3a}$. *Ans.* $\frac{15ax-12a}{30bc-42c}$.

40. Divide $\left(3a+\frac{2x-1}{3}\right)$ by $\frac{4a}{5n}$. *Ans.* $\frac{45an+10nx-5n}{12a}$.

41. Divide $\left(4x-\frac{3a+1}{5}\right)$ by $\frac{3a}{7n}$.

Ans. $\frac{140nx-21an-7n}{15a}$.

42. Divide $\left(2a-\frac{3b+4c-5}{4}\right)$ by $\left(3x+\frac{4y-2a+7}{3}\right)$.

Ans. $\frac{24a-9b-12c+15}{36x+16y-8a+28}$.

43. Reduce $\dfrac{\frac{3a}{5b}}{\frac{6x}{7y}}$ to a simple fraction. *Ans.* $\dfrac{7ay}{10bx}$.

44. Reduce $\dfrac{x-\frac{3a}{5}}{y+\frac{4c}{7}}$ to a simple fraction. *Ans.* $\dfrac{35x-21a}{35y+20c}$.

45. Reduce $\dfrac{x+\frac{2a-3}{3}}{a-\frac{2x+1}{4}}$ to a simple fraction.

Ans. $\dfrac{12x+8a-12}{12a-6x-3}$.

SECTION IX.

SIMPLE EQUATIONS, OR EQUATIONS OF THE FIRST DEGREE.

Ex. 1. CHARLES and Henry together have 45 marbles, and Henry has 4 times as many as Charles; how many marbles has each?

Analysis. Suppose we say that Charles has *one* boxful; then, as Henry has 4 times as many, he has 4 boxes; then together they have 5 boxes; and if 5 boxes contain 45 marbles, then 1 box will contain $\frac{1}{5}$ of 45 marbles, which will be 9 marbles, which equals Charles's share, and 4 boxes will contain 36, which will be Henry's share.

We might express our problem algebraically, thus:

Let 1 box = Charles's share;
then, 4 boxes = Henry's share;
then, 1 box + 4 boxes = 45 marbles,
or, 5 boxes = 45 marbles,
and 1 box = $\frac{45}{5}$ = 9 marbles, Charles's share;
4 boxes = 4 × 9 = 36 marbles, Henry's share.

But instead of using the word *boxes*, suppose we make use of the letter

x to represent Charles's marbles; then $4x$ will represent Henry's marbles, and we shall have

$$\begin{aligned} & x+4x=45, \\ \text{or,} \quad & 5x=45, \\ \text{and} \quad & x=\tfrac{45}{5}=\ 9, \text{ Charles's marbles.} \\ & 4x=4\times 9=36, \text{ Henry's marbles.} \end{aligned}$$

148. The expression $x+4x=45$ is what is called an EQUATION.* Hence,

An EQUATION *is an algebraic expression representing the equality between two quantities, connected by the sign of equality.*

The two quantities connected by the sign = are called the MEMBERS of the equation; the quantities on the *left* constituting the FIRST member, and those on the *right* the SECOND member.

They are sometimes called SIDES of the equation, the first being called the *left-hand* side, and the second the *right-hand* side. The quantities connected by the signs + and − are called the TERMS of the equation. Thus, in the equation $a+x=c-d$, $a+x$ is the *first* member, $c-d$ is the *second* member; a and x are the *terms* of the first member, and c and d are the terms of the second member.

149. The EQUATION is the distinguishing feature of Algebra. A very large part of the investigations and reasonings in Algebra are conducted by means of equations.

150. In all researches by means of equations it is a fundamental principle always to keep the two sides of the equation equal to each other. To do this we must remember that *whenever we make changes on one side we must make exactly the same changes on the other side;* or, in other words,

1. If we *add* any quantity to one member, we must add an exactly equal quantity to the other member.

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* *Equation*, from the Latin word *æquo, to make equal.*

2. If we *subtract* any quantity from one member, we must *subtract* an equal quantity from the other.

3. If we *multiply* one member by any quantity, we must *multiply* the other member by an equal quantity.

4. If we *divide* one member by any quantity, we must *divide* the other member by an equal quantity.

Ex. 2. A farmer has 360 acres of land planted in wheat, corn and cotton. He has 3 times as many acres in corn as in wheat, and *twice* as many acres in cotton as in corn. How many acres has he of each?

Analysis. Let $x =$ the number of acres of wheat;
then, $3x =$ " " " corn,
and $6x =$ " " " cotton.

Then, as the number of acres of wheat, corn and cotton together make 360 acres, we have the equation

$$x+3x+6x=360.$$

Adding the x's together, we have

$$10x=360,$$

$$x=\tfrac{360}{10}=36=\text{acres of wheat},$$

and $3x=3\times 36=108=$ " corn,
$6x=6\times 36=216=$ " cotton.

Proof. $36+108+216=360.$

When we began the solution of this problem we did not *know* how many acres were in wheat, but we represented the number of acres by x; this letter therefore represented an *unknown* quantity; and as we progressed with our work we found that 10 *times* this *unknown* quantity was equal to 360, and therefore the *unknown* quantity was $\frac{1}{10}$ of 360, which gave 36 acres for the amount in wheat.

151. UNKNOWN QUANTITIES are usually represented by the *last* letters of the alphabet, as x, y, z, etc. KNOWN QUANTITIES are expressed by *numerals*, or by the *first* letters of the alphabet, as a, b, c, etc.

NOTE.—Algebra differs from Arithmetic in making use of *unknown* quantities. These are introduced into calculation, and are added, subtracted, multiplied, etc. precisely as if they were *known* or their values were *given*.

3. A shepherd has 4 flocks of sheep, numbering altogether 396 sheep. The second flock contains twice as many sheep as the first; the third has as many as the first and second together; and the fourth has twice as many as the third. How many sheep in each flock?

Analysis. Let x = number in the first flock;
then $2x$ = " " second "
$3x$ = " " third "
$6x$ = " " fourth "

And since all the flocks added together make 396 sheep, we have the equation,

$$x+2x+3x+6x=396.$$

Now, adding the x's together (41), we have

$$12x=396;$$

then $x=\frac{396}{12}=33$ = sheep in first flock.
$2x=2\times33=66$ = " second "
$3x=3\times33=99$ = " third "
$6x=6\times33=198$ = " fourth "

Proof. $33+66+99+198=396$.

In this problem we did not at first *know* how many sheep were in the first flock, but we represented this *unknown* number by x; and as we proceeded we found x to be equal to a number that we *did know*—viz. 33. In this case we are said to *have found the value of the unknown quantity.*

In solving problems the great object is to find the *value of the unknown quantity;* and this is done by bringing the *unknown* quantity to stand *by itself* on one side of the equation, and the *known* quantities to stand on the opposite side. This is generally called *reducing an equation.* Hence,

152. *The* REDUCTION OF AN EQUATION *consists in bringing the unknown quantity to stand by itself on one side, and the known quantities to stand on the other side.*

4. A sportsman went out gunning, and killed a number of doves, robins and partridges, the whole number being 56. There were 3 times as many robins as doves, and as

many partridges as doves and robins together; how many of each sort of birds did he kill?

Ans. 7 doves; 21 robins; 28 partridges.

5. A farmer has in his pasture 63 head of cattle, consisting of cows, calves and sheep. There are twice as many calves as cows, and twice as many sheep as calves; how many has he of each sort?

Ans. 9 cows; 18 calves; 36 sheep.

6. Charles, James and Thomas together have 150 chestnuts. James has twice as many as Charles, and Thomas has 3 times as many as Charles; how many has each?

Ans. Charles, 25; James, 50; Thomas, 75.

7. Three men, A, B and C, united to build a schoolhouse costing 1750 dollars; of which sum B contributed 3 times as much as A, and C gave twice as much as B; what sum was contributed by each?

Ans. A, 175 dolls.; B, 525 dolls.; C, 1050 dolls.

8. A farmer invested 240 dollars in the purchase of goats, hogs and sheep. The sum paid for the hogs was 3 times that paid for the goats, and the sheep cost as much as the hogs and goats together; what was the cost of each kind?

Ans. Goats, 30 dolls.; hogs, 90 dolls.; sheep, 120 dolls.

9. Four men, A, B, C and D, invested 1512 dollars in trade; of which B put in twice as much as A; C contributed as much as A and B together; and D put in twice as much as C; what sum was put in by each?

Ans. A, 126 dolls.; B, 252 dolls.; C, 378 dolls.; D, 756 dolls.

10. A merchant has two pieces of cassimere which together measure 72 yards, and one of the pieces is 6 yards longer than the other; what is the length of each piece?

Analysis. Let $x =$ No. of yards in the shorter piece;
then, $x+6 =$ " " " longer "

And as the two pieces together measure 72 yards, we have the equation,

$$x+x+6=72,$$

or, $$2x+6=72. \quad (A)$$

Now, let us *subtract* 6 from *both* sides of this equation, which will not destroy the equality of the two members (Ax. 2), and we have the equation,

$$2x+6-6=72-6.$$

Now, as we have $+6$ and -6 on the same side of the equation, they exactly balance each other, and therefore they may be dropped out or *cancelled*. Our equation will then stand,

$$2x=72-6, \quad (B)$$

or, $$2x=66,$$

and $$x=33=\text{No. yards in shorter piece},$$

$$x+6=33+6=39=\text{" " longer "}$$

After subtracting 6 from both sides of our equation, we obtained the equation $2x=72-6$ (marked B). This equation is the same as our original equation, $2x+6=72$ (marked A), *except* that $+6$ has been transferred to the other side, and has received a contrary sign.

This change is called TRANSPOSITION.* Hence,

153. TRANSPOSITION *is the operation of carrying a quantity from one side of an equation to the other, at the same time changing the sign.*

11. Henry and James have 40 marbles between them, and James has 4 less than Henry; how many marbles has each?

Analysis. Let $x=$ Henry's marbles,
then $x-4=$ James's "

And as the two together have 40 marbles, we have the equation,

$$x+x-4=40,$$

or, $$2x-4=40. \quad (A)$$

Now, let us *add* $+4$ to both sides, which will not destroy the equality (Ax. 1), and we have the equation,

$$2x-4+4=40+4.$$

Since we have $+4$ and -4 on the same side of the equation, they *just*

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* *Transposition,* from the Latin *transpono, to put across, to carry over.*

balance each other, and therefore may be dropped. Our equation will then stand,

$$2x = 40 + 4. \quad \text{(B)}$$

or, $2x = 44,$

$x = 22 =$ Henry's marbles,

and $x - 4 = 22 - 4 = 18 =$ James's marbles.

Ans. Henry, 22 marbles; James, 18 marbles.

We would again call attention to the fact that after *adding* $+4$ to both sides of the equation, the resulting equation, $2x = 40 + 4$ (B), was exactly the same as our original equation, $2x - 4 = 40$ (A), except that -4 has been transferred to the other side, and its sign has been changed. After dividing by the coefficient of x, we have x, the *unknown* quantity, standing *by itself* on one side, and the known quantity standing on the other; and thus we know the *value* of x, because we have found that it is equal to a *known* quantity.

Hence,

154. *When known quantities are connected with the unknown quantity by the sign + or −, the equation is reduced by* TRANSPOSING *the known quantities to the other side, and changing the sign.*

12. A farmer has two flocks of sheep, numbering in all 145 sheep, and one flock has 21 more sheep than the other; how many sheep in each flock?

Analysis. Let $x =$ No. in the smaller flock;
then, $x + 21 =$ " " larger "

and as the two flocks together number 145 sheep, we have the equation,

$$x + x + 21 = 145.$$

Uniting the x's, $2x + 21 = 145.$

Now, transposing 21 to the other side, we have

$$2x = 145 - 21,$$

or, $2x = 124,$

$x = \frac{124}{2} = 62 =$ sheep in smaller flock,
$x + 21 = 62 + 21 = 83 =$ " larger "

13. John and Henry together have 125 chestnuts, and Henry has 15 more than John; how many chestnuts has each? *Ans.* John, 55 chestnuts; Henry, 70 chestnuts.

14. A farmer has 156 acres planted in corn and cotton, and there are 22 acres more of cotton than of corn; how many acres of each?

Ans. Corn, 67 acres; cotton, 89 acres.

15. A and B started on a journey, and together they walked 78 miles, but B walked 8 miles farther than A; how far did each walk? *Ans.* A, 35 miles; B, 43 miles.

16. A merchant has two pieces of muslin, measuring together 96 yards, and one piece contains 6 yards more than the other; how many yards in each piece?

Ans. 45 yards; 51 yards.

17. Richard and Thomas together caught 160 minnows, but Richard caught 10 more than Thomas; how many minnows did each catch?

Ans. Richard, 85 minnows; Thomas, 75 minnows.

18. A grocer has three jugs which together hold 36 quarts. The second contains 5 quarts more than the first, and the third 5 quarts more than the second; how many quarts does each contain?

Analysis. Let x = No. of quarts in 1st,
then, $x+5$ = " " 2d,
$x+10$ = " " 3d.

And as the three together contain 36 quarts, we have the equation,

$$x+x+5+x+10=36.$$

Uniting terms, $3x+15=36$;
transposing 15, $3x=36-15$;
or, $3x=21$,
$x=\frac{21}{3}=7$ = No. quarts in 1st,
$x+5=7+5=12$ = " " 2d,
$x+10=7+10=17$ = " " 3d.

Proof. $7+12+17=36$.

19. A merchant has 3 pieces of silk which together measure 128 yards. The second piece is 6 yards longer than the

first, and the third is 8 yards longer than the second; what is the length of each piece?

Ans. 1st, 36 yards; 2d, 42 yards; 3d, 50 yards.

20. A, B and C made a contribution amounting to 95 cents. B put in 5 cents more than A, and C gave 4 cents more than B; how much did each contribute?

Ans. A, 27 cents; B, 32 cents; C, 36 cents.

21. A man bought a buggy, a horse and a wagon for 400 dollars. The horse cost 75 dollars more than the wagon, and the buggy cost 25 dollars more than the horse; what was the cost of each?

Ans. Wagon, 75 dolls.; horse, 150 dolls.; buggy, 175 dolls.

22. A farmer has 3 bins of wheat containing 375 bushels. The second contains 35 bushels more than the first, and the third contains 20 bushels more than the second; how much does each contain? *Ans.* 1st, 95 bu.; 2d, 130 bu.; 3d, 150 bu.

23. Thomas, William and James went on a fishing-excursion, and caught 159 perch. William caught 13 more than Thomas, and James 7 more than William; how many perch did each catch?

Ans. Thomas, 42; William, 55; James, 62.

24. A farmer invested 265 dollars in goats, hogs and sheep. The hogs cost 20 dollars more than the goats, and the sheep cost 30 dollars more than the hogs; what amount did he invest in each?

Ans. Goats, 65 dolls.; hogs, 85 dolls.; sheep, 115 dolls.

25. What number is that which, multiplied by 4 and diminished by 20, gives the same result as twice the number increased by 10?

Analysis. Let x = the number;
then $4x$ = the number multiplied by 4,
$2x$ = twice the number.

Then, according to the conditions, we have

$$4x - 20 = 2x + 10.$$

Here we have the *unknown* quantity on *both* sides of the equation. When this is the case it must be *transposed*, just as in the case of *known* quantities.

Transposing the *unknown* quantity to the left and the *known* quantity to the right, we have

$$4x-2x=20+10.$$

Uniting terms, $2x=30,$

$$x=\tfrac{30}{2}=15=\text{number}.$$

Proof. 4 times $15=60$, and $60-20=40$;
Twice $15=30$, and $30+10=40$.

26. What number is that which, multiplied by 3 and diminished by 4, gives the same result as when multiplied by 2 and increased by 6? *Ans.* 10.

27. What number increased by 25 will equal 3 times the number diminished by 13? *Ans.* 19.

28. What number increased by 36 will equal 4 times the number diminished by 27? *Ans.* 21.

29. A gentleman being asked his age, replied that 20 times his age, diminished by 20, was equal to 19 times his age, increased by 19. What was his age? *Ans.* 39.

155. When the *same term* with the same sign is on *opposite sides* of the equation, it may be cancelled or dropped from each side without the trouble of transposing. Thus, if we have the equation,

$$4x+a-5=2x+a+25,$$

as a with the *same* sign is on opposite sides of the equation, we can simply drop it from both, and we then have

$$4x-5=2x+25.$$

Then, transposing, $4x-2x=25+5$;
uniting terms, $2x=30$

$$x=\tfrac{30}{2}=15.$$

This method of dropping out equal quantities from opposite sides is frequently very convenient, as it saves the necessity of actual transposition.

156. *All* the terms of an equation may be transposed from one side to the other. Each side of the equation will then become equal to 0. Thus, in the equation

$$5x - 8 = 3x + 4,$$

if we transpose *all* the terms on the *right* over to the left, we have

$$5x - 3x - 8 - 4 = 0.$$

To show that the left side of this equation is equal to 0, we have only to find the *value* of x in the original equation, and substitute this value for x in the last equation. Taking our first equation,

$$5x - 8 = 3x + 4,$$

transposing terms, $5x - 3x = 8 + 4,$

uniting terms, $2x = 12,$

$$x = 6.$$

Now, substituting this value of x for x in the equation above, we have

$$30 - 18 - 8 - 4 = 0,$$

or, $30 - 30 = 0,$

or, $0 = 0.$

30. What number is that whose *half* and whose *third* part added together make 40?

Analysis. Let $x =$ the number;

then $\frac{x}{2} =$ half the number,

and $\frac{x}{3} =$ one-third of the number.

Then, as its half and its third are to be equal to 40, we have the equation,

$$\frac{x}{2} + \frac{x}{3} = 40.$$

Now, let us multiply both sides of the equation by 6, the product of the denominators of the fractions, and we have

$$3x + 2x = 240.$$

Uniting terms, $5x = 240,$

$$x = \tfrac{240}{5} = 48.$$

Proof. $\frac{1}{2}$ of 48 is 24; $\frac{1}{3}$ of 48 is 16. Then $24+16=40$, which fulfils the requirements of the question.

In multiplying both sides of the equation by 6 we did not affect the equality of the sides, because we *increased* each side equally (Ax. 3).

Hence,

157. *An equation may be cleared of fractions by multiplying each side by the product of all the denominators.*

31. A tree was cut down, and $\frac{1}{3}$ of its length added to $\frac{1}{4}$ of it measured 42 feet; how long was the tree?

Ans. 72 feet.

32. One-fourth and one-fifth of the number of sheep in a pasture are 54 sheep; how many are in the pasture?

Ans. 120 sheep.

33. John has a box of chestnuts, and $\frac{1}{3}$ of the number $+\frac{1}{7}$ of the number make 120 chestnuts; how many chestnuts has he? *Ans.* 252 chestnuts.

34. A gentleman being asked his age, replied, "$\frac{1}{3}$ of my age $+\frac{1}{8}$ of it is equal to $27\frac{1}{2}$ years;" what was his age?

Ans. 60 years.

35. One-half and one-fifth of the number of trees in an orchard number 126 trees; how many trees in the orchard?

Ans. 180 trees.

36. Of the animals in a certain pasture, $\frac{1}{4}$ are cows, $\frac{1}{3}$ are goats and the remaining 25 are sheep; how many animals are in the pasture?

Analysis. Let $x=$ No. in the pasture,

$\frac{x}{4}=$ No. of cows,

$\frac{x}{3}=$ No. of goats,

$25=$ No. of sheep.

Now, as the cows, the goats and the sheep make up all the animals in the pasture, we have the equation,

$$x=\frac{x}{4}+\frac{x}{3}+25.$$

Clearing of fractions, by multiplying each side of the equation by 12, the product of all the denominators, we have

$$12x = 3x + 4x + 300,$$

or, $$12x = 7x + 300.$$

Transposing, $$12x - 7x = 300;$$

uniting terms, $$5x = 300,$$

$$x = \tfrac{300}{5} = 60.$$

Proof. $\frac{1}{4}$ of 60 is 15, $\frac{1}{3}$ of 60 is 20, and $15+20+25=60$, or $60=60$.

37. A post is $\frac{1}{5}$ in the earth, $\frac{1}{3}$ in the water and 7 feet out of the water; how long is the post? *Ans.* 15 feet.

38. Of the pupils in a certain school, $\frac{1}{4}$ study Geometry, $\frac{1}{3}$ study Algebra, $\frac{1}{5}$ learn Latin, and the remaining 13 learn Arithmetic; how many pupils in the school?

Ans. 60 pupils.

39. A farmer plants $\frac{1}{5}$ of his land in corn, $\frac{1}{4}$ in wheat, $\frac{1}{3}$ in oats, and the remainder, which is 26 acres, he plants in cotton; how many acres does he plant? *Ans.* 120 acres.

40. Of the trees in an orchard, $\frac{1}{8}$ bear cherries, $\frac{1}{6}$ bear figs, $\frac{1}{3}$ bear pears, and the remaining 54 produce peaches; how many trees in the orchard? *Ans.* 144 trees.

41. A merchant invested $\frac{1}{3}$ of his money in dry goods, $\frac{1}{4}$ in groceries, $\frac{1}{6}$ in hats, and the remaining 324 dollars in shoes; what was the whole amount of his investment?

Ans. 1296 dollars.

42. What number is that a *sixth* part of which exceeds its *eighth* part by 12?

Analysis. Let $x =$ the number,

$\frac{x}{6} =$ a sixth part,

$\frac{x}{8} =$ an eighth part.

Then, as the *difference* between a *sixth* part and an *eighth* part is 12, we have the equation,

$$\frac{x}{6} - \frac{x}{8} = 12.$$

Now, clearing of fractions, by multiplying each side of the equation by 24, which is the *least common multiple* of all the denominators, we have

$$4x-3x=288.$$

Uniting terms, $x=288.$

Proof. $\frac{1}{6}$ of 288 is 48, and $\frac{1}{8}$ of 288 is 36, and 48 *exceeds* 36 by 12.

An equation may *always* be cleared of fractions by multiplying each side by the *product of all the denominators;* but it is often much more convenient simply to multiply each side by the *least common multiple* of all the denominators, just as we did in this last example. We recommend the pupil to use the *least common multiple* whenever it can be done.

43. What number is that $\frac{3}{4}$ of which exceeds $\frac{2}{3}$ of the same by 18? *Ans.* 216.

44. Four-fifths of a certain number exceeds $\frac{2}{3}$ of it by 26; what is the number? *Ans.* 195.

45. If $\frac{5}{7}$ of a certain number is less than $\frac{4}{5}$ of it by 24, what is the number? *Ans.* 280.

46. If $\frac{5}{6}$ of a certain number exceeds $\frac{7}{9}$ of it by 20, what is the number? *Ans.* 360.

47. A gentleman being asked his age, replied, "If you add $\frac{1}{4}$ of my age to $\frac{1}{5}$ of it, and increase this sum by $\frac{1}{6}$ of my age, the result will be 37 years;" what was his age?

Analysis. Let $x=$ his age,

$\frac{x}{4}=\frac{1}{4}$ of his age,

$\frac{x}{5}=\frac{1}{5}$ " "

$\frac{x}{6}=\frac{1}{6}$ " "

Then, as $\frac{1}{4}$ and $\frac{1}{5}$ and $\frac{1}{6}$ of his age added together make 37 years, we have the equation,

$$\frac{x}{4}+\frac{x}{5}+\frac{x}{6}=37.$$

Clearing of fractions by multiplying each side by 60, the *least common multiple* of the denominators, we have

$$15x+12x+10x=2220.$$

Uniting terms, $37x=2220,$

$$x=\tfrac{2220}{37}=60=\text{his age.}$$

Proof. $\frac{1}{4}$ of 60 is 15, $\frac{1}{5}$ of 60 is 12, and $\frac{1}{6}$ of 60 is 10, and $15+12+10=37$.

48. What number is that the sum of the third part, the fourth part and the fifth part of which is 235? *Ans.* 300.

49. One-half, one-third, one-fourth and one-fifth of a certain number added together make 231; what is the number? *Ans.* 180.

50. What number is that of which the sum of the third part, the fifth part and the tenth part is 95? *Ans.* 150.

51. There is a certain number of which if $\frac{2}{3}$ be added to $\frac{3}{5}$, and this sum be increased by $\frac{5}{9}$ of the number, the result will be 164; what is the number? *Ans.* 90.

52. A merchant, after selling 24 yards from a roll of bagging, found that $\frac{3}{4}$ of the remainder measured 36 yards; how many yards were originally in the roll?

Analysis. Let $x=$ No. of yards at first,
$24=$ " " cut off;
then $(x-24)=$ " " remaining,
and $\frac{3}{4}(x-24)=\frac{3}{4}$ of what was left.

Now, since $\frac{3}{4}$ of what remained measured 36 yards, we have the equation,

$$\tfrac{3}{4}(x-24)=36.$$

Multiplying the two factors on the left side of the equation, we have

$$\frac{3x-72}{4}=36.$$

Clearing of fractions, $3x-72=144;$

transposing, $3x=144+72,$

$$3x=216,$$

$$x=\tfrac{216}{3}=72.$$

Proof. $72-24=48$; then $\frac{3}{4}$ of 48 is 36.

53. A farmer sold 54 bushels out of his bin of wheat, and then found that $\frac{2}{3}$ of the remainder measured 128 bushels; how much was in the bin at first? *Ans.* 246 bushels.

54. From the top of a May-pole a piece 18 feet long was broken off by the wind, when it was found that $\frac{5}{6}$ of the portion that was left was 45 feet in length; how long was the pole at first? *Ans.* 72 feet.

55. Thirty-six dollars were taken from a purse, when it was found that $\frac{5}{7}$ of what remained was equal to 90 dollars; how much was in the purse at first? *Ans.* 162 dollars.

56. From a cask of wine 14 gallons were drawn, when it was found that $\frac{7}{8}$ of what was left in the cask measured 35 gallons; how much was in the cask at first?.

Ans. 54 gallons.

57. A shepherd, after losing by disease 18 out of his flock of sheep, and then selling $\frac{3}{4}$ of the remainder, had 27 sheep left; how many sheep had he at first?

Analysis. Let $x =$ No. at first,
$18 =$ No. lost by disease:
then, $x - 18 =$ No. remaining,
and $\frac{3}{4}(x-18) = \frac{3x-54}{4} =$ No. he sold.

Now, it is plain that the number he *lost* and the number he *sold*, with the number that finally *remained*, must have made up his *whole* flock, Hence, we have the equation,

$$x = 18 + \frac{3x-54}{4} + 27.$$

Clearing of fractions by multiplying by 4, the denominator, we have

$$4x = 72 + 3x - 54 + 108;$$

transposing, $4x - 3x = 72 - 54 + 108$;
uniting terms, $x = 126 =$ No. of sheep.

Proof. After losing 18 he had 108 left; and then selling $\frac{3}{4}$ of 108, which is 81, he had 27 left.

58. After spending 51 dollars of my money for board, and

investing $\frac{2}{3}$ of the remainder in clothing, I had 36 dollars left; how much had I at first? *Ans.* 159 dollars.

59. A farmer planted 45 acres of his land in wheat, $\frac{3}{5}$ of the remainder in cotton, and the remaining 96 acres in corn; of how many acres did his farm consist? *Ans.* 285 acres.

60. A merchant went to Atlanta to purchase goods. After spending $756 of his money for groceries, he invested $\frac{5}{7}$ of the remainder in dry goods, and then had $336 left; what was his money originally? *Ans.* $1932.

61. A carpenter cut off at one time 12 feet from a stick of timber, and at another time $\frac{7}{9}$ of the remainder, and still had 6 feet left; what was the original length of the stick? *Ans.* 39 feet.

158. The operations to which our attention has been directed in this section are classed under the head of the SOLUTION OF PROBLEMS.

The pupil will have noticed that the *solution of a problem* consists of *two* important steps.

I. The *first* step is to translate the question from *common* language into *algebraic* language; or, in other words, to put it into the form of an equation. This is usually known as *stating the problem.*

II. The *second* step is to *reduce the equation* after the *problem is stated,* so that the *unknown* quantity shall stand *by itself* on one side and the *known* quantities on the other.

This latter process, as the pupil must have already noticed, involves *four* different operations:

1. *Clearing the equation of fractions;*

2. *Transposing the unknown quantities to one side, and the known quantities to the other;*

3. *Uniting the terms;*

4. *Dividing by the coefficient of the unknown quantity.*

62. John can do a job of work in 6 days, and Richard can do the same work in 9 days; in how many days could they finish it by working together?

Analysis. Let $x =$ No. of days.

The work done may be regarded as *one* job, and hence may be represented by *unity*, or 1. Now, as John can do the job in 6 days, then in *one* day he can do $\frac{1}{6}$ of it, and in x days he can do x *times* as much $= \frac{x}{6}$ part of the work.

Again: As Richard can do the job in 9 days, then in *one* day he can do $\frac{1}{9}$ of it, and in x days he can do x *times* as much, or $\frac{x}{9}$ part of the work.

But the part done by John + the part done by Richard is equal to the *whole* job; hence we have the equation,

$$\frac{x}{6}+\frac{x}{9}=1.$$

Clearing of fractions by multiplying by 18, the *least common multiple* of the denominators (**157**), we have

$$3x+2x=18,$$

or, $5x=18,$

and $x=\frac{18}{5}=3\frac{3}{5}$ days. *Ans.* $3\frac{3}{5}$ days.

63. If A can build a house in 12 days, and B can build it in 16 days, in what time can they build it by working together? *Ans.* $6\frac{6}{7}$ days.

64. If C can reap a field of wheat in 5 days, and D can reap it in 8 days, in what time can they reap it if they work together? *Ans.* $3\frac{1}{13}$ days.

65. If Charles can plough a field in 10 days, and Henry can plough the same field in 15 days, in how many days can they plough it by working together? *Ans.* 6 days.

66. If John can mow a meadow in 8 days, Richard in 10 days and James in 12 days, in what time can they finish it by mowing together? *Ans.* $3\frac{9}{37}$ days.

67. If A can lay out a flower-garden in 5 days, B in 6

days and C in 10 days, in how many days can they do it by working together? *Ans.* $2\frac{1}{7}$ days.

68. If one carpenter can build a barn in 20 days, another carpenter in 30 days, and a third carpenter in 40 days, in how many days could they finish it if they work together? *Ans.* $9\frac{3}{13}$ days.

69. A father divides a farm of 504 acres between his two sons, Richard and Henry, in the ratio of 3 to 4; what is the share of each?

Analysis. Let $x =$ Richard's share.
It is plain that when Richard receives 3 acres, Henry receives 4, and when Richard receives *one* acre (*i. e.* $\frac{1}{3}$ of the previous amount), Henry receives also $\frac{1}{3}$ of his previous amount, or $\frac{4}{3}$ of an acre; and when Richard receives x acres, Henry receives $\frac{4x}{3}$ acres. But the two together receive 504 acres; hence we have the equation,

$$x+\frac{4x}{3}=504.$$

Clearing of fractions, $3x+4x=1512$;
uniting terms, $7x=1512$;
dividing, $x=\frac{1512}{7}=216$ acres, Richard's share,
$\frac{4x}{3}=\frac{4\times216}{3}=288$ " Henry's share.

We might solve this problem in another way:

Analysis. Let $x =$ Henry's share.
Now, it is obvious that as often as Henry receives 4 acres, Richard receives 3; and when Henry receives *one*, Richard receives $\frac{3}{4}$; and when Henry receives x, Richard receives $\frac{3x}{4}$. Hence we have the equation, $x+\frac{3x}{4}=504.$

Clearing of fractions, $4x+3x=2016$;
uniting terms, $7x=2016$;
dividing, $x=\frac{2016}{7}=288=$ Henry's share,
$\frac{3x}{4}=\frac{3\times288}{4}=216=$ Richard's share.

NOTE. – It will be time profitably employed for the teacher to show his pupils that the same problem may be solved by different methods. The effect will be to exercise their inventive faculty and awaken their enthusiasm.

70. A gentleman divided \$6048 between his son and daughter, in the ratio of 5 to 7; what was the share of each? *Ans.* Son, \$2520; daughter, \$3528.

71. Two men, A and B, formed a partnership in business, and gained \$1820, which they divided in the ratio of 5 to 9; what was the share of each?

Ans. A, \$650; B, \$1170.

72. A mixture was made of 84 gallons of wine and brandy, the quantity of each being in the ratio of 2 to 5; how much of each was put in? *Ans.* Wine, 24 gal.; brandy, 60 gal.

73. Two men, C and D, invested \$3564 in the purchase of a steam-engine, the amount furnished by each being in the ratio of 5 to 6; how much did each furnish?

Ans. C, \$1620; D, \$1944.

74. Charles and Reuben gathered 918 chestnuts, which they divided in the ratio of 7 to 10; what was the share of each?

Ans. Charles, 378 chestnuts; Reuben, 540 chestnuts.

75. Three men, A, B and C, built a schoolhouse for \$1524, and the money furnished by each was in the ratio of the numbers 3, 4 and 5; what sum did each furnish?

Analysis. Let $x =$ sum furnished by A.

Now, when A furnishes 3 dollars, B furnishes 4; and when A furnishes *one* dollar, B furnishes $\frac{4}{3}$; and when A furnishes x dollars, B furnishes $\frac{4x}{3}$ dollars.

Again: When B furnishes 4 dollars, C furnishes 5; and when B furnishes *one* dollar, C furnishes $\frac{5}{4}$; but B (as we have just shown) furnishes $\frac{4x}{3}$ dollars; therefore, C furnishes $\frac{4x}{3} \times \frac{5}{4}$ dollars $= \frac{20x}{12}$

dollars $=\frac{5x}{3}$ dollars. Hence, as the three furnish 1524 dollars, we have the equation,

$$x+\frac{4x}{3}+\frac{5x}{3}=1524.$$

Clearing of fractions, $3x+4x+5x=4572;$

uniting terms, $12x=4572;$

dividing, $x=\frac{4572}{12}=381$, A's amount.

$\frac{4x}{3}=\frac{4\times 381}{3}=508$, B's "

$\frac{5x}{3}=\frac{5\times 381}{3}=635$, C's "

We might also solve this example by taking x to represent B's money, or we might take x to represent C's money; and it would be a profitable exercise for the young student to work the example by each of the latter methods.

76. John, William and Charles gathered 1545 walnuts, which they divided in the ratio of 4, 5 and 6; how many walnuts did each receive?

Ans. John, 412 walnuts; William, 515 walnuts; Charles, 618 walnuts.

77. A farmer has three bins of wheat, which together hold 315 bushels, and the quantities in the several bins are in the ratio of the numbers 5 and 7 and 9; what quantity is contained in each bin?

Ans. 1st, 75 bushels; 2d, 105 bushels; 3d, 135 bushels.

78. A merchant has 3 pieces of muslin, measuring together 152 yards, and the several lengths are in the ratio of 5 and 6 and 8; what is the length of each piece?

Ans. 1st, 40 yards; 2d, 48 yards; 3d, 64 yards.

79. Two travellers, A and B, 210 miles apart, travel toward each other until they meet. A's speed is 6 miles per hour, and B's 4 miles; how far does each travel before the two meet?

Analysis. Let x = No. of hours before meeting.

Now, as A goes 6 miles in *one* hour, then in x hours he will go x *times* as far, which will be $6x$ miles.

Again: As B travels 4 miles an hour, in x hours he will travel x *times* as far, which will be $4x$ miles; and as the two together travel 210 miles, we have the equation,

$$6x+4x=210.$$

Uniting terms, $$10x=210,$$

$$x=\frac{210}{10}=21 \text{ hours}.$$

As A goes 6 miles an hour, his whole distance will be $21\times6=126$ miles, and B's distance $=21\times4=84$ miles.

Ans. A, 126 miles; B, 84 miles.

80. Two men, C and D, 968 miles apart, travel on a railroad until they meet, C progressing at the rate of 20 miles an hour, and D at 24 miles an hour; how far will each travel before the two meet?

Ans. C, 440 miles; D, 528 miles.

81. Two men set out to travel until they meet, one leaving Augusta at the same time that the other leaves Atlanta, the distance between the two places being 171 miles. The first goes 4 miles, and the second 5 miles an hour. How far will each travel before meeting the other?

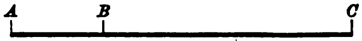
Ans. First, 76 miles; second, 95 miles.

82. Two locomotives at the extremities of a railroad 480 miles long run toward each other—one at the rate of 22 miles an hour, and the second at 26 miles an hour; how far will each run before meeting the other?

Ans. First, 220 miles; second, 260 miles.

83. A railroad-train running 18 miles an hour is followed by another train 36 miles behind, running at the rate of 22 miles an hour; in how many hours will the second train overtake the first?

Analysis.



Suppose the foremost train to be at B, and the hindmost at A, and that they come together at the point C; then it is plain that the foremost train runs the distance from B to C and the hindmost train from A to C; and the hindmost train must run 36 miles more than the foremost.

Let x = No. of hours required.

Then, as the foremost train runs 18 miles in *one* hour, in x hours it will run x *times* as far, which will be $18x$ miles; and the hindmost train, running 22 miles in *one* hour, in x hours will run $22x$ miles; hence,

$18x$ = No. miles run by foremost train,
$22x$ = " " hindmost "

And as the hindmost train runs 36 miles *more* than the foremost, we must *add* 36 miles to the distance run by the foremost to make it equal to that run by the hindmost; hence, we have the equation,

$$22x = 18x + 36.$$

Transposing, $22x - 18x = 36$;
uniting, $4x = 36$;
dividing, $x = \frac{36}{4} = 9$ hours.

Proof. In 9 hours the foremost train will run $18 \times 9 = 162$ miles, and the hindmost train $22 \times 9 = 198$ miles; then, $198 = 162 + 36$.

It will be noticed that in this problem, although the point sought after was *time*, yet we formed our equation out of *miles*, and by means of this equation we were enabled to find the value of x, the *unknown* quantity, which gave us the *time*.

The student will not fail to notice another distinguishing feature, which is this: that in the formation of an equation the two sides or members always consist of things of the *same name* or *denomination*. Thus, in this last problem we made *miles* equal *miles*.

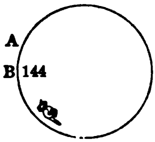
It is *always* the fact that the two sides of an equation represent quantities of the *same denomination;* and hence it follows that

An EQUATION *is nothing more than different expressions for the same quantity.*

84. A locomotive running 30 miles an hour had 90 miles the start of another locomotive pursuing at the rate of 36

miles an hour; in how many hours will the second overtake the first? *Ans.* 15 hours.

85. A and B start to walk in the same direction around a circular park 1080 feet in circumference. B is 144 feet ahead of A, and walks at the rate of 150 feet in a minute, whilst A follows at the rate of 162 feet in a minute; in how many minutes will A overtake B? *Ans.* 12 minutes.



86. Suppose the conditions to remain the same as in the last problem, except that B is ahead of A by just *half the circumference* of the circle; in how many minutes will they be together? *Ans.* 45 minutes.

87. The conditions remaining the same as to speed, after A has overtaken B, as in the last problem, how long will it be before he overtakes him again? *Ans.* 90 minutes.

NOTE.—Let the student remember that A must gain a *whole* circumference before they can be together again.

88. What number is that which being increased by $\frac{3}{5}$ of itself, and diminished by 17, will give 103?

Analysis. Let x = the number;

then, $\frac{3x}{5}$ = three fifths of the number;

and by the conditions of the question we have

$$x+\frac{3x}{5}-17=103.$$

Clearing of fractions, $5x+3x-85=515$;

transposing, $5x+3x=515+85$;

uniting, $8x=600$,

$$x=\frac{600}{8}=75=\text{number}.$$

Proof. $\frac{3}{5}$ of 75 is 45; and 45 *added* to 75 makes 120; and 120 *diminished* by 17 leaves 103.

89. What number increased by $\frac{4}{7}$ of itself, and diminished by 25, will leave 151? *Ans.* 112.

90. What number diminished by $\frac{2}{9}$ of itself, and increased by 24, will be equal to 136? *Ans.* 144.

91. What number diminished by $\frac{4}{11}$ of itself, and increased by 18, will give 102? *Ans.* 132.

92. What number increased by $\frac{5}{6}$ of itself, and diminished by 44, will leave 44? *Ans.* 48.

93. What number diminished by $\frac{5}{9}$ of itself, and the remainder multiplied by 7, will give 476?

Analysis. Let $x =$ the number,

$$\frac{5x}{9} = \text{five ninths of the number,}$$

$$\left(x - \frac{5x}{9}\right) = \text{the remainder;}$$

and by the conditions of the problem we have

$$\left(x - \frac{5x}{9}\right) \times 7 = 476.$$

Actually multiplying, we have $7x - \frac{35x}{9} = 476$;

clearing of fractions, $63x - 35x = 4284$;

uniting, $28x = 4284$;

dividing $x = \frac{4284}{28} = 153 = \text{number.}$

Proof. $\frac{5}{9}$ of 153 is 85; $153 - 85 = 68$; and $68 \times 7 = 476$.

94. What number diminished by $\frac{2}{7}$ of itself, and the remainder multiplied by 9, will give 810? *Ans.* 126.

95. What number increased by $\frac{2}{9}$ of itself, and this sum multiplied by 8, will give 704? *Ans.* 72.

96. What number increased by $\frac{2}{5}$ of itself, and this result multiplied by 9 and then diminished by 20, will give 1681? *Ans.* 135.

97. What number diminished by $\frac{3}{8}$ of itself, and this result multiplied by 5 and then increased by 25, will give 425? *Ans.* 128.

98. What number diminished by $\frac{2}{9}$ of itself, and the remainder divided by 7, will give 24 as the result? *Ans.* 216.

99. A farmer bought an equal number of goats, sheep, hogs and cows for \$510. Each goat cost \$2; each sheep, \$3; each hog, \$4; and each cow \$25; what number of each did he buy?

Analysis. Let $x =$ the number of each.

As *one* goat cost \$2, then x goats cost x *times* as much, etc.

Therefore, $2x =$ cost of the goats,

$3x =$ " " sheep,

$4x =$ " " hogs,

$25x =$ " " cows;

and by the conditions of the question, we have

$$2x+3x+4x+25x=510;$$

uniting $34x=510;$

dividing, $x=15=$ number of each sort.

100. A farmer invested \$180 in lambs, kids, pigs and calves, buying an equal number of each. For each lamb he gave \$1; for each kid, \$2; for each pig, \$3; and for each calf, \$4; how many of each did he buy? *Ans.* 18.

101. A grocer invested \$13.44 in oranges, bananas, pineapples and cocoanuts, buying an equal number of each. He gave 4 cents apiece for the oranges, 3 cents apiece for the bananas, 12 cents apiece for the pineapples, and 9 cents apiece for the cocoanuts; how many of each did he buy? *Ans.* 48.

102. A confectioner invested \$26.50 in walnuts, almonds, raisins and candy, buying an equal number of pounds of each. He gave 13 cents a pound for the walnuts, 14 cents for almonds, 11 cents for raisins and 15 cents for candy; how many pounds did be buy of each? *Ans.* 50 pounds.

103. Divide the number 75 into two such parts that 3 times the greater may exceed 4 times the less by 15.

Analysis. Let $x =$ the greater;

then, $(75-x) =$ the less,

$3x =$ *three* times the greater,

$4(75-x) =$ *four* " " less.

Then, as 3 times the greater *exceeds* 4 times the less by 15, it is plain that if we *add* 15 to 4 times the less, the result will be equal to 3 times the greater. Hence, we have the equation,

$$3x=4(75-x)+15.$$

Actually multiplying, $3x=300-4x+15;$

transposing, $3x+4x=300+15;$

uniting, $7x=315,$

$$x=45=\text{greater},$$

$$75-x=30=\text{less}.$$

Proof. 3 times $45=135$; 4 times $30=120$; and 135 *exceeds* 120 by 15.

We might have obtained another equation in this way: It is obvious that if we *subtract* 15 from 3 times the greater, it will give *four* times the less. Hence, we have

$$3x-15=4(75-x),$$

or, $3x-15=300-4x;$

transposing, $3x+4x=300+15;$

uniting, $7x=315,$

$$x=45=\text{greater},$$

$$75-x=30=\text{less}.$$

Or we might have obtained a *third* equation in this way: Since the *difference* between *three* times the greater and *four* times the less is 15, we have the equation,

$$3x-4(75-x)=15,$$

or $3x-300+4x=15;$

transposing, $3x+4x=300+15;$

uniting, $7x=315,$

$$x=45=\text{greater},$$

$$75-x=30=\text{less}.$$

In this last equation the student will notice that the sign $-$ is prefixed to the quantity $4(75-x)$. This *minus* sign indicates that the *entire* quantity within the parenthesis is to be *subtracted*, which is done by *changing all the signs* of that quantity.

NOTE.—As we have elsewhere stated, so we repeat, that it will be a most valuable mental training for the pupil to exercise his ingenuity in working the same problem by different methods. He may not advance quite so rapidly, but his grasp of the subject will be much more thorough. In Algebra, as in some other things, "the race is not always to the swift."

104. Two men shared 170 dollars in such a manner that 4 times the greater share exceeded 5 times the less share by 5 dollars; what was the share of each?

Ans. Greater, 95 dollars; less, 75 dollars.

105. John and Henry divided 540 chestnuts between themselves, so that twice the share of John was greater than 3 times that of Henry by 40; what was the share of each? *Ans.* John, 332 chestnuts; Henry, 208 chestnuts.

106. Two men starting from opposite points travelled altogether 277 miles, and on meeting it was discovered that 3 times the distance of the first was greater by 15 miles than 5 times the distance of the second; how far did each travel?

Ans. First, 175 miles; second, 102 miles.

107. After paying out $\frac{1}{4}$ of my money for board and $\frac{1}{5}$ of it for railroad-fare, I had 99 dollars left; how much had I at first?

Analysis. Let $x =$ my money at first,

$\frac{x}{4} =$ the money paid for board,

$\frac{x}{5} =$ " " " " railroad-fare.

Now, it is obvious that the money paid for board, + the money paid for railroad fare, + the money left, must equal the money originally on hand. Hence, we have the equation,

$$x=\frac{x}{4}+\frac{x}{5}+99.$$

Clearing of fractions, $20x=5x+4x+1980,$

$20x=9x+1980;$

transposing, $20x-9x=1980;$

uniting, $11x=1980,$

$x=180$ dollars.

Proof. $\frac{1}{4}$ of $180=45$; $\frac{1}{5}$ of $180=36$; then, $45+36+99=180$.

108. Charles gave $\frac{2}{5}$ of his chestnuts to his sister and $\frac{1}{3}$ of them to his little brother, and then had 120 chestnuts left; how many chestnuts had he at first?

Ans. 450 chestnuts.

109. A merchant invested $\frac{1}{4}$ of his money in dry goods, $\frac{1}{5}$ in hardware, $\frac{1}{6}$ in shoes, and the remainder, amounting to \$598, in groceries; what sum had he at first? *Ans.* \$1560.

110. A farmer planted $\frac{1}{5}$ of his land in corn, $\frac{1}{3}$ in cotton, $\frac{1}{7}$ in wheat, and the remainder, which was 136 acres, he devoted to pasturage; of how many acres did his farm consist?
Ans. 420 acres.

111. Of the trees in an orchard, $\frac{3}{8}$ of them bear peaches, $\frac{3}{16}$ produce apples, $\frac{1}{4}$ bear pears, $\frac{3}{32}$ produce cherries, and the remaining 54 are plum trees; how many trees are in the orchard? *Ans.* 576 trees.

112. A farmer had two flocks of sheep, each containing the same number. From one of these he sold 36, and from the other 63, and found just *twice* as many remaining in one as in the other; of how many sheep did each flock consist?

Analysis. Let $x =$ No. in each flock at first,
$x - 36 =$ No. left in first flock.
$x - 63 =$ “ “ second flock.

Now, it is plain that the flock from which he sold the most must have had the *fewest* left; therefore, by the conditions,

$$2(x-63) = x - 36,$$

or, $$2x - 126 = x - 36;$$

transposing, $$2x - x = 126 - 36;$$

uniting, $$x = 90 \text{ sheep}.$$

Proof. $90 - 36 = 54$; $90 - 63 = 27$; and 54 is *twice* 27.

113. A merchant had 2 hogsheads of molasses, each holding the same number of gallons. From one of these he drew 42, and from the other 24 gallons, and then found just 3 times as many gallons remaining in one as in the other; how many gallons were in each hogshead originally? *Ans.* 51 gallons.

114. Two military companies contained the same number of men. From one of these 53 men were detailed for a special service, and 35 from the other, when 4 times as

many were left in the one as in the other; of how many men did each company consist? *Ans.* 59 men.

115. A merchant had 3 pieces of cassimere of *equal* length. From the first of these pieces he cut 15 yards, from the second 19 yards, and from the third 22 yards, and found that the three remnants together measured 52 yards; what was the length of each piece? *Ans.* 36 yards.

116. A gentleman divided a dollar among 9 children, giving some of them 10 cents apiece, and to the others 12 cents each; how many children were there of each class?

Analysis. Let $x =$ No. he gave 12 cents each,
$(9-x) =$ No. " 10 "
$12x =$ whole sum given 1st class,
$10(9-x) =$ " " 2d "

Then, as he gave $1, or 100 cents, to both classes, we have the equation,

$$12x+10(9-x)=100,$$

or, $12x+90-10x=100;$
transposing, $12x-10x=100-90;$
uniting, $2x=10,$
$x=5=$ 1st class,
$9-x=4=$ 2d "

Proof. 5 at 12 cents $=$ 60 cents; 4 at 10 cents $=$ 40 cents. Then, $60+40=100$.

117. A grocer bought 75 oranges for 2 dollars and 75 cents, paying 5 cents apiece for a part of them, and 3 cents apiece for the remainder; how many of each price did he buy? *Ans.* 50 at 3 cents; 25 at 5 cents.

118. A man travelled 225 miles, partly by railroad and partly by stage-coach, paying $11.25 for his fare. On the railroad he paid 4 cents a mile, and by coach 7 cents per mile; how many miles did he travel by each mode of conveyance? *Ans.* 150 miles by railroad; 75 miles by coach.

119. A grocer bought a barrel filled with apples and oranges, 315 in all. For the oranges he paid 5 cents

apiece, and for the apples 2 cents apiece, and the whole cost of the apples was the same as that of the oranges; how many of each were there?

Ans. 225 apples; 90 oranges.

120. A confectioner bought a lot of oranges and pineapples, 286 in all, the oranges at 4 cents apiece and the pineapples at 10 cents apiece; and the whole cost of the pineapples was 3 *times* that of the oranges; how many of each did he buy? *Ans.* 156 pineapples; 130 oranges.

121. A father is 5 times as old as his son, but in 6 years the father will be only 3 times the age of the son; what is the age of each?

Analysis. Let $x =$ son's age,
$5x =$ father's age.

Then, in 6 years each will be 6 years older than at present; hence,

$x+6=$ son's age 6 years hence,
$5x+6=$ father's " " "

Then, by the conditions of the question the father's age will be 3 times the son's age; therefore,

$$5x+6=3(x+6),$$

or, $5x+6=3x+18;$
transposing, $5x-3x=18-6;$
uniting, $2x=12,$
$x=6=$ son's age,
$5x=30=$ father's age.

Proof. In 6 years the father will be 36, and the son 12; and 36 = 3 times 12.

122. A gentleman at the time of his marriage was 3 times as old as his wife, but after they had lived together 16 years he was only twice as old; what were their ages at the time of their marriage? *Ans.* Husband, 48; wife, 16.

123. William is twice as old as his brother Henry, but 6 years ago William was 3 times as old; what are their ages?

Ans. William, 24; Henry, 12.

124. A lady is 36 years of age, and her daughter is 8 years old; in how many years will the mother be just 3 times as old as her daughter? *Ans.* 6 years.

125. Divide the number 36 into *three* such parts that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second and $\frac{1}{4}$ of the third shall be equal to each other.

Analysis. Since $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second and $\frac{1}{4}$ of the third are equal to *each other*, it is obvious that they are each equal to the *same thing;* therefore, x may represent either $\frac{1}{2}$ of the first part, $\frac{1}{3}$ of the second or $\frac{1}{4}$ of the third. Then, if $x=\frac{1}{2}$ of the first part, the *whole* of the first part will be $2x$; and if $x=\frac{1}{3}$ of the second part, the *whole* of the second will equal $3x$; and if $x=\frac{1}{4}$ of the third part, then $4x=$ the *whole* of the third. Hence, we have

$$\begin{aligned} 2x &= \text{first part,} \\ 3x &= \text{second ``} \\ 4x &= \text{third ``} \end{aligned}$$

and as the three parts make up 36, we have

$$2x+3x+4x=36;$$

uniting, $$9x=36,$$

$$x=4.$$

Then, $$\begin{aligned} 2x &= 8 = \text{first part,} \\ 3x &= 12 = \text{second ``} \\ 4x &= 16 = \text{third ``} \end{aligned}$$

126. Divide 48 into three such parts that $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second and $\frac{1}{5}$ of the third shall be equal to each other. *Ans.* 12, 16 and 20.

127. Divide 65 into three such parts that $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second and $\frac{1}{6}$ of the third shall be equal to each other. *Ans.* 15, 20 and 30.

128. Divide 112 into four such parts that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, $\frac{1}{4}$ of the third and $\frac{1}{5}$ of the fourth shall be equal to each other. *Ans.* 16, 24, 32 and 40.

129. A farmer wishes to mix corn-meal worth 63 cents a bushel with oat-meal worth 36 cents, so that he may have a

mixture of 81 bushels, worth 48 cents a bushel; how many bushels of each must he take?

Analysis. Let $x =$ No. of bu. of corn-meal,
$(81-x) =$ " " " oat-meal,
$63x =$ *whole* value of the corn-meal,
$36(81-x) =$ *whole* " " oat-meal;

and 81 bushels (the mixture) at 48 cents per bushel will be worth $81 \times 48 = 3888$ cents; and as the values of the corn-meal and oat-meal together are to be the same as the value of the mixture, we have

$$63x + 36(81-x) = 3888,$$

or, $$63x + 2916 - 36x = 3888;$$

transposing, $$63x - 36x = 3888 - 2916;$$

uniting, $$27x = 972,$$

$$x = 36, \text{ No. bu. corn-meal,}$$

$$(81-x) = 45, \quad \text{" oat-meal.}$$

Proof. 36 bushels at 63 cents = \$22.68
45 " " 36 " = 16.20
81 " \$38.88

and 81 bushels (mixture) at 48 cents = \$38.88.

The problem we have just solved would come under the Rule of *Alligation alternate*, and could be worked by the methods given in Arithmetic. The *arithmetical* method is complicated and tedious, while the *algebraic* plan is shorter and more easily understood. Such examples illustrate the great utility of Algebra, and its great superiority, in many respects, over Arithmetic.

130. A grocer mixes brown sugar worth 10 cents a pound with clarified sugar worth 15 cents, so as to make a barrel containing 220 pounds, worth 13 cents per pound; how much of each kind must he take?

Ans. Brown, 88 pounds; clarified, 132 pounds.

131. A vintner wishes to mix malaga wine worth 60 cents a gallon with sherry at 95 cents, so as to make a barrel of 35 gallons, worth 80 cents a gallon; how many gallons of each must he take? *Ans.* Malaga, 15 gals.; sherry, 20 gals.

132. A merchant has two kinds of tea, one worth 60 cents a pound, the other worth 90 cents; how many pounds of each kind must he take to form a box of 75 pounds which shall be worth 70 cents per pound?

Ans. 50 pounds at 60 cents; 25 pounds at 90 cents.

133. What number is that from which if 12 be subtracted, $\frac{5}{7}$ of the remainder will be 60?

Analysis. Let $x =$ the number,
then $(x-12) =$ the remainder,
$\frac{5}{7}(x-12) = \frac{5}{7}$ of the remainder.

Then, by the conditions of the question, we have

$$\tfrac{5}{7}(x-12)=60,$$

or,
$$\frac{5x-60}{7}=60.$$

Clearing of fractions, $5x-60=420$;
transposing, $5x=420+60$,
$$5x=480,$$
$$x=96.$$

Proof. 12 from 96 leaves 84, and $\frac{5}{7}$ of $84=60$.

134. What number is that from which if 18 be subtracted, $\frac{4}{9}$ of the remainder will be 24? *Ans.* 72.

135. A gentleman having spent 24 dollars, finds that $\frac{5}{6}$ of what he has left is 80 dollars; how much had he at first? *Ans.* 120 dollars.

136. What number is that to which if 25 be added, $\frac{5}{9}$ of the sum will be 50? *Ans.* 65.

137. A farmer had a certain number of sheep, and after buying 44 more found that $\frac{7}{12}$ of his entire flock was 154 sheep; how many had he at first? *Ans.* 220 sheep.

138. It is required to divide the number 84 into two such parts that the less being divided by the greater the quotient shall be $\frac{3}{4}$.

Analysis. Let $x =$ the greater part,
$84-x =$ the less "

Then, by the conditions of the question, we have

$$\frac{84-x}{x}=\frac{3}{4}.$$

Clearing of fractions, $336-4x=3x;$

transposing, $336=4x+3x;$

uniting $336=7x,$

$48=x=\text{greater},$

$36=84-x=\text{less}.$

Proof. $\frac{36}{48}=\frac{3}{4}$.

In this example we find that x stands on the *right-hand* side of the equation. Most usually, the *unknown* quantity stands finally on the *left*. This, however, is a matter of indifference; if $48=x$, then, necessarily, $x=48$.

We might have solved this problem thus:

Let $x=$ the less part,

then, $(84-x)=$ the greater part,

and $\frac{x}{84-x}=\frac{3}{4}.$

Clearing of fractions, $4x=252-3x;$

transposing, $4x+3x=252,$

$7x=252,$

$x=36=\text{less part},$

$(84-x)=48=\text{greater part}.$

139. Divide 90 into two such parts that the quotient of the less, divided by the greater, shall be $\frac{2}{3}$.

Ans. 36 and 54.

140. It is required to divide 108 into two such parts that the greater being divided by the less, the quotient may be $3\frac{1}{2}$.

Ans. 24 and 84.

141. A cistern holds 648 gallons of water. It is filled in 18 minutes by 2 pipes, one of which conveys 6 gallons more per minute than the other; how many gallons does each pipe convey per minute?

Analysis. Let $x=$ No. gallons per minute in smaller pipe,

$(x+6)=$ " " " larger "

Then, $18x=$ *whole* No. gallons conveyed by smaller,

$18(x+6)=$ " " " larger.

And as the two pipes together fill the cistern, we have

$$18x+18(x+6)=648,$$

or, $18x+18x+108=648,$

$$36x+108=648,$$

$$36x=648-108,$$

$$36x=540,$$

$$x=15 \text{ gal.}=\text{smaller pipe},$$

$$x+6=21 \text{ “ } =\text{larger “}$$

142. A railroad-tank holding 1378 gallons is filled in 26 minutes by two pipes, one of which conducts 5 gallons less per minute than the other; how many gallons does each pipe convey? *Ans.* 24 and 29 gallons.

143. A cistern holding 1136 gallons is filled in 16 minutes by three pipes, the second of which conveys twice as many gallons per minute as the first, and the third conveys 6 gallons more than the second; how many gallons does each pipe convey per minute? *Ans.* 13, 26 and 32 gallons.

144. At a certain election 1237 votes were cast, and if the successful candidate had received 23 votes more, he would have *doubled* the votes of his opponent; how many votes did each receive?

Analysis. Let $x=$ No. of votes of successful candidate,

$1237-x=$ “ “ unsuccessful “

If the successful candidate had received 23 more votes, he would then have had $(x+23)$ votes, and he would have *doubled* his opponent; hence, we have the equation,

$$x+23=2(1237-x),$$

or, $x+23=2474-2x;$

transposing, $x+2x=2474-23;$

uniting, $3x=2451,$

$$x=817=\text{No. votes of successful candidate},$$

$$(1237-x)=420=\text{ “ “ unsuccessful “}$$

Proof. If the candidate elected had received 23 votes more, he would have had $817+23=840$; and 840 is the *double* of 420.

NOTE.—We might have solved this problem with equal ease by letting x represent the votes of the unsuccessful candidate.

145. At a certain town-election 1512 votes were cast, and the candidate elected lacked only 20 votes of receiving just 3 times as many votes as his opponent; how many votes did each receive? *Ans.* Suc. can., 1129; unsuc. can., 383.

146. At an election for major of a battalion 545 votes were polled, and the officer chosen was elected by only 5 votes over his opponent; how many votes were cast for each?
Ans. 270 and 275.

147. Divide 60 into two such parts that if the greater be divided by 5, and the less by 3, the difference of the quotients shall be 4.

Analysis. Let $x =$ the greater part,
$(60-x) =$ " less "
$\frac{x}{5} =$ quotient of greater divided by 5,
$\frac{60-x}{3} =$ " " less " " 3.

Then, by the conditions of the problem, we have

$$\frac{x}{5} - \frac{60-x}{3} = 4.$$

Clearing of fractions, $3x - 300 + 5x = 60$;
transposing, $3x + 5x = 300 + 60$,
$8x = 360$,
$x = 45 =$ greater,
$60 - x = 15 =$ less.

The student will notice in this problem that the sign $-$ is prefixed to the fraction $\frac{60-x}{3}$; this sign indicates that the value of the *entire* fraction is to be *subtracted* from $\frac{x}{5}$; and this is done by *changing all the signs* of the numerator; hence, when cleared of fractions the equation stands,

$$3x - 300 + 5x = 60.$$

It must be remembered that the $-$ sign affects the *whole* of the numerator, and *not* merely the $-x$; it affects the 60 as well as the x. The sign of 60 is actually $+$, and when changed becomes $-$, and the $-x$ becomes $+x$.

This being a very important point, and one about which students are apt to make great mistakes, we shall give a few examples more fully to impress it:

148. Given $4x-\frac{3x-6}{3}=9+\frac{x+1}{2}$, to find the value of x.

Clearing of fractions, by multiplying by 6, the least common multiple, we have $24x-6x+12=54+3x+3$;
transposing, $24x-6x-3x=54+3-12$;
uniting, $15x=45$,
$x=3$.

Notice here that the $-$ sign came before the fraction $\frac{3x-6}{3}$, showing that the *whole* of the quotient was to be *subtracted* from $4x$, and therefore we *changed all the signs* of the numerator.

We repeat again:

Let the student remember that whenever the $-$ sign comes before a fraction, the signs of ALL *the terms of the numerator must* ALWAYS *be changed.*

149. Given $x+\frac{4x-5}{5}=10-\frac{2x-4}{3}$, to find x. *Ans.* $x=5$.

150. Given $2x+\frac{x-3}{3}+5=20-\frac{x+4}{5}$, to find x.

Ans. $x=6$.

151. Given $\frac{4x}{7}+\frac{x-1}{2}-2=7-\frac{5x-7}{14}$, to find x.

Ans. $x=7$.

152. Given $2x-\frac{3x-6}{3}=12-\frac{4x-4}{2}$, to find x.

Ans. $x=4$.

It will be a good exercise for the pupil to *prove* that he has obtained the *true* value of the letter which represents the unknown quantity. This he may do by substituting the numerical value for the letter itself in the *original* equation. If the two sides of the equation are then equal, the value found for the unknown quantity is correct.

Thus, in the last example, if we substitute 4 for x in the original equation, we shall have

$$8-\frac{12-6}{3}=12-\frac{16-4}{2},$$

or, $$8-2=12-6,$$

or, $$6=6.$$

153. Given $x+\frac{3x+1}{5}=2x-\frac{3x-3}{7}$, to find x. *Ans.* $x=8$

154. Given $\frac{x+1}{2}-\frac{x+1}{3}=3-\frac{x+3}{4}$, to find x.

Ans. $x=5$.

155. Given $x-\frac{x+2}{7}=\frac{4x+2}{5}$, to find x. *Ans.* $x=12$.

The teacher will find it greatly to the advantage of the student to require him to *translate* algebraic equations like the last into *common* language. This exercise prepares him to translate problems from *common* language into the *algebraic* form; or, in other words, fits him for *stating* ordinary problems algebraically.

Our last example might be translated thus:

Find a number such that this number, diminished by the quotient of the number increased by 2 *and divided by* 7, *may be equal to the quotient of* 4 *times the number increased by* 2 *and divided by* 5.

156. Given $\frac{3x+4}{5}-\frac{2x+4}{6}=\frac{x-1}{3}$, to find x. *Ans.* $x=7$.

157. Given $3x-\frac{2x+3}{3}=9-\frac{7x+3}{8}$, to find x. *Ans.* $x=3$.

158. Given $\frac{15-3x}{3}+\frac{10+x}{4}=8-\frac{3x+4}{5}$, to find x.

Ans. $x=2$.

159. Given $\frac{14-2x}{6}+\frac{6+4x}{5}=6-\frac{3+x}{2}$, to find x.

Ans. $x=1$.

160. Given $\frac{3x-3}{5}+6=16-\frac{4x-3}{3}$, to find x. *Ans.* $x=6$.

161. Given $x+\frac{2x-4}{4}=2x+1-\frac{2x+1}{3}$, to find x.

Ans. $x=10$.

162. Given $\frac{x-1}{8}+\frac{x-3}{6}+2=x-\frac{2x-3}{3}$, to find x.

Ans. $x=9$

163. Given $\frac{7-x}{4}+\frac{9+2x}{3}-2=2x-\frac{x+1}{2}$, to find x.

Ans. $x=3$.

164. Given $\frac{x}{5}-a=\frac{x}{6}+2$, to find x. *Ans.* $x=30a+60$.

165. Given $\frac{bc}{x}=ac+b+\frac{3}{x}$, to find x. *Ans.* $x=\frac{bc-3}{ac+b}$.

166. Given $2x-b+ax=\frac{a+x}{4}$, to find x. *Ans.* $x=\frac{a+4b}{7+4a}$.

167. A drayman promised to deliver 30 flower-vases, agreeing to receive 9 cents apiece for all that he delivered safely, and to pay 15 cents for all that he broke. On settlement he received one dollar and fifty cents; how many did he break and how many did he deliver safely?

Analysis. Let $x=$ No. delivered safely,
$(30-x)=$ " broken,
$9x=$ what he received,
$15(30-x)=$ what he forfeited.

And as the drayman must have received a sum of money equal to the *difference* between $9x$ and $15(30-x)$, we have

$$9x-15(30-x)=150,$$

or, $$9x-450+15x=150,$$

$$9x+15x=450+150,$$

$$24x=600,$$

$$x=25=\text{No. unbroken,}$$

$$30-x=5=\text{No. broken.}$$

168. A gentleman hired a laborer for 20 days, agreeing to pay him 75 cents for every day he worked, but for every day that he was idle he was to forfeit 50 cents for his board. At the end of the time he received $8.75; how many days did he work and how many was he idle?

Ans. Worked 15 days; idle 5 days.

LITERAL EQUATIONS.

159. In the preceding problems the *known* quantities have been represented by *numbers*, and the answers have been given in *numbers*. These are called NUMERICAL EQUATIONS.

Equations, however, sometimes occur in which some or all of the known quantities are represented by *letters;* these are called LITERAL EQUATIONS.

We will illustrate by a few examples:

Ex. 1. Henry has 315 chestnuts, which he wishes to put into 3 boxes, so that the second box may contain twice as many chestnuts as the first, and the third twice as many chestnuts as the second; how many must he put in each box?

Analysis. Let $x=$ No. in first box,

$2x=$ " second box,

$4x=$ " third "

Then, $$x+2x+4x=315,$$

$$7x=315,$$

$$x=45=\text{No. in first,}$$

$$2x=90=\text{'' second,}$$

$$4x=180=\text{'' third.}$$

We will now form a *literal* equation after the model of the *numerical* equation just given:

2. Thomas has n chestnuts, which he wishes to put into 3 boxes, so that the second may contain twice as many chestnuts as the first, and the third twice as many as the second; how many must he put in each box?

Analysis. Let $x =$ No. in first box,
$2x =$ " second box,
$4x =$ " third "

Then, $x + 2x + 4x = n$,
$7x = n$,
$x = \frac{n}{7} =$ No. in first,
$2x = \frac{2n}{7} =$ " second,
$4x = \frac{4n}{7} =$ " third.

From the preceding operation we see that the first box must contain $\frac{1}{7}$ of the entire number of chestnuts, the second must contain $\frac{2}{7}$ of the whole, and the third $\frac{4}{7}$ of the whole.

Suppose n to represent 336 chestnuts; then,

$\frac{n}{7} = \frac{336}{7} =$ 48 chestnuts = first box,
$\frac{2n}{7} = \frac{672}{7} =$ 96 " = second box,
$\frac{4n}{7} = \frac{1344}{7} = 192$ " = third "

Or, suppose n to represent 392 chestnuts;

then, $\frac{n}{7} = \frac{392}{7} = 56 =$ No. in first,
$\frac{2n}{7} = \frac{784}{7} = 112 =$ " second,
$\frac{4n}{7} = \frac{1568}{7} = 224 =$ " third.

We see, then, that this formula will answer for ten thousand different questions of the *same sort* as the one just solved. This process is called GENERALIZATION; and when we solve a problem finding the unknown quantity equal to a certain form of *known letters*, we are said to *generalize the problem.*

3. A farmer has 3 bins of wheat containing 312 bushels. The second contains twice, and the third 3 times, as much as the first; how many bushels in each bin?

Ans. 1st, 52 bu.; 2d, 104 bu.; 3d, 156 bu.

The following is a *generalization* of the preceding problem:

4. Divide the number p into three such parts that the second may be m times as great as the first, and the third n times as great as the first.

Ans. 1st, $\frac{p}{1+m+n}$; 2d, $\frac{mp}{1+m+n}$; 3d, $\frac{np}{1+m+n}$.

5. A man being asked his age, replied that if the third part and the fourth part of his age were added to his age, the sum would be 76 years; what was his age?

Ans. 48 years.

A *generalization* of this problem would be about as follows:

6. What number is that whose mth part and nth part, being added to the number itself, will give p?

Analysis. Let $x=$ the number,

$\frac{x}{m}=m$th part,

$\frac{x}{n}=n$th "

Then, by the conditions we have

$$x+\frac{x}{m}+\frac{x}{n}=p.$$

Clearing of fractions, $mnx+nx+mx=mnp$;

resolving into factors, $(mn+n+m)x=mnp$;

dividing, $x=\frac{mnp}{mn+n+m}$.

7. A farmer planted $\frac{1}{4}$ of his land in wheat, $\frac{1}{5}$ in corn,

and the remaining 88 acres he devoted to pasturage; of how many acres did his farm consist? *Ans.* 160 acres.

The preceding problem generalized:

8. What number is that whose mth part, added to its nth part, and this increased by q, shall be equal to the number itself? *Ans.* $\frac{mnq}{mn-n-m}$.

9. What number is that the fourth part of which is greater than its seventh part by 27? *Ans.* 252.

The preceding problem generalized:

10. What number is that the mth part of which is greater than its nth part by d? *Ans.* $\frac{mnd}{n-m}$.

11. The sum of two numbers is 16, and their difference is 4; what are the numbers?

Let $x=$ the less number,

$x+4=$ the greater number.

The sum of these is equal to 16; hence,

$$2x+4=16,$$

$$2x=12,$$

$x=6$, the less number,

$x+4=10$, the greater number.

The preceding generalized:

12. The sum of two numbers is s, and their difference is d; what are the numbers?

Let $x=$ the less number,

$x+d=$ the greater number.

Then, as the sum of these is equal to s, we have

$$2x+d=s,$$

$$2x=s-d,$$

$x=\frac{s-d}{2}=\frac{s}{2}-\frac{d}{2}=$ the less number,

and $x+d=\frac{s}{2}-\frac{d}{2}+d=\frac{s}{2}-\frac{d}{2}+\frac{2d}{2}=\frac{s}{2}+\frac{d}{2}=$ the greater number.

In the answer to this last problem we have reached an important general truth, which may be thus expressed:

Half the difference of two quantities added to half their sum is equal to the greater; and half their difference subtracted from half their sum is equal to the less.

13. The sum of two numbers is 96, and their difference is 16; what are the numbers? *Ans.* 40 and 56.

14. The sum of two numbers is a, and their difference is b; what are the numbers? *Ans.* ——.

The student cannot have failed to notice that in solving problems *two* distinct processes are employed:

I. To *translate* the question from *common* into *algebraic* language; or, in other words, to put it in the form of an equation.

II. To *reduce* the equation; *i. e.* to bring the unknown quantity to stand *by itself* on one side and the known quantities on the other.

No definite rule can be given for translating a problem from common into algebraic language, or for *stating* the problem, as it is usually called. This can be learned only by frequent practice. We will only say this much to the student:

Represent the unknown quantity sought after by x, or one of the last letters of the alphabet; then carefully examine the language of the problem, sentence by sentence, so as to discover the relation which the several quantities bear to each other; then express these relations by algebraic signs.

The *second* part of the process of solving problems—viz. *reducing the equation*—can be learned by specific and definite rules. The student has already been doing this within the last several pages, and he has seen that it consists of *four* distinct steps, which we again briefly recapitulate:

I. *Clear the equation of fractions;*

II. *Transpose the known quantities to one side, and the unknown quantity to the other;*

III. *Unite the quantities, and reduce them to their simplest forms;*

IV. *Divide each side by the coefficient of the unknown quantity.*

PROMISCUOUS PROBLEMS.

1. Find a number such that the sum of its half, its third and its fourth part shall be 91. *Ans.* 84.

2. Find a number the sum of whose mth part and its nth part shall be equal to b. *Ans.* $\frac{bmn}{m+n}$.

3. A young man being asked his age, replied, "If you add my father's age, which is 42 years, to twice my age, the sum will be 4 times my age;" what was his age? *Ans.* 21 years.

4. What number is that $\frac{4}{5}$ of which is greater than $\frac{3}{7}$ of it by 26? *Ans.* 70.

5. A confectioner bought 150 oranges for $6.60, paying 4 cents apiece for a portion of them, and 5 cents apiece for the others; how many of each sort did he buy?
Ans. 90 at 4 cents; 60 at 5 cents.

6. What number is that whose mth part and its nth part, being added to the number itself, will give a number equal to c? *Ans.* $\frac{cmn}{mn+n+m}$.

7. A merchant had two rolls of bagging, each containing the same number of yards. Having sold 42 yards from one piece and 24 from the other, he found twice as many yards remaining in the one as in the other; how many yards were in each piece? *Ans.* 60 yards.

8. What number is that $\frac{3}{4}$ of which is as much *above* 45 as $\frac{1}{2}$ of the number is *below* 45? *Ans.* 72.

9. A merchant had two pieces of silk, each containing the same number of yards. Having cut (p) yards from one and (d) yards from the other, he finds 3 times as many yards in the latter as in the former; how many yards were in each piece? *Ans.* $\frac{3p-d}{2}$.

10. A man bought a horse and buggy for \$245, and $\frac{1}{4}$ of the price of the buggy was equal to $\frac{1}{3}$ of the price of the horse; what was the price of each?

Ans. Buggy, \$140; horse, \$105.

11. James can do a piece of work in 7 days, and Thomas can do the same in 9 days; in what time can they finish it by working together? *Ans.* $3\frac{15}{16}$ days.

12. A man paid \$5 with silver quarters and dimes, there being 29 pieces of money; how many pieces were there of each sort? *Ans.* 14 quarters; 15 dimes.

13. A gentleman paid \$195 for his horse and saddle, and the horse cost $5\frac{1}{2}$ times as much as the saddle; what was the cost of each? *Ans.* Horse, \$165; saddle, \$30.

14. Divide 120 into two such parts that the quotient of the greater divided by the less may be $6\frac{1}{2}$.

Ans. Greater, 104; less, 16.

15. Out of a cask of brandy from which $\frac{1}{4}$ part had leaked 24 gallons were afterward drawn, when the cask was found to be half full; how much did the cask hold?

Ans. 96 gallons.

16. Three merchants, A, B and C, entered into a speculation, for which they subscribed \$2016; B paid three times as much as A, and C paid as much as A and B together; how much did each pay? *Ans.* A, \$252; B, \$756; C, \$1008.

17. It is required to find such a number that its third part, increased by its fourth part, shall be equal to the number itself diminished by 15. *Ans.* 36.

18. A merchant has $675 in his safe, consisting of greenbacks and silver. The amount in greenbacks is $3\frac{1}{2}$ times as much as the silver; what is the amount of each?

Ans. Greenbacks, $525; silver, $150.

19. A gentleman dying left $875 to be divided among his three servants, in such a way that the first was to receive double what the second received, and the second double the portion of the third; what was each servant's share?

Ans. First, $500; second, $250; third, $125.

20. Two persons, 243 miles apart, set out to meet each other, one going 4 miles an hour, while the other goes 5 miles; how far will each have travelled when they meet?

Ans. 108 miles; 135 miles.

21. A gentleman left $2863 to be divided among his two sons and a daughter, in such a way that the oldest son was to receive $300 more than the daughter, and the daughter $200 more than the younger son; what did each receive?

Ans. Elder son, $1221; daughter, $921; younger son, $721.

22. A farmer began business with a flock of sheep. The first year he lost $\frac{1}{6}$ of them by disease, but afterward bought 20 sheep, and then found that he had just 100 sheep; how many had he at first? *Ans.* 96 sheep.

23. A farmer has 56 animals, consisting of horses, cows and sheep, in a pasture. There are 6 more cows than horses, and the sheep are 12 more in number than the horses and cows together; how many of each are there?

Ans. 8 horses; 14 cows; 34 sheep.

24. A father's age is triple that of his son, but 11 years hence it will be double; what is the age of each?

Ans. Father, 33; son, 11.

25. Divide 42 into two parts, so that 4 times the greater part may exceed 5 times the less by 6; what are the two parts? *Ans.* 24 and 18.

26. A merchant has 3 pieces of silk, which together measure 158 yards. The second is 10 yards longer than the first, and the third is 12 yards longer than the second; what is the length of each piece?

Ans. 42, 52 and 64 yards.

27. A man being asked his age, replied, "If you divide my age by 5, the quotient will be equal to 40 diminished by the quotient of 3 times my age, divided by 9;" what was his age? *Ans.* 75 years.

28. John, William and Henry together have 80 apples. John has 15 more than William and 10 more than Henry; how many has each?

Ans. John, 35; William, 20; Henry, 25.

29. A detachment belonging to an army was composed of infantry, cavalry and artillery. The infantry were $\frac{5}{7}$ of the whole, the cavalry were 50 *more* than $\frac{1}{6}$ of the whole, and the artillery were 30 *less* than $\frac{1}{10}$ of the whole; how many men were in the detachment? *Ans.* 1050 men.

30. Divide 54 into three such parts that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second and $\frac{1}{4}$ of the third shall all be equal to each other. *Ans.* 12, 18 and 24.

31. A father is now 3 times as old as his son; in 21 years the father's age will be just double the son's age; how old is each? *Ans.* Father, 63; son, 21.

32. A man being asked how many sheep he had, replied, "If I had as many more, and half as many more, and two sheep and a half, I would then have 150;" how many sheep had he? *Ans.* 59 sheep.

33. A man gained 20 dollars in a trade, and then lost $\frac{1}{4}$ of the entire sum he then had, when he found that he had 135 dollars left; how much had he at first?

Ans. 160 dollars.

34. A cask was filled with a mixture of brandy, wine and

water. The brandy was $\frac{2}{3}$ of the whole, the wine was 5 gallons more than $\frac{1}{6}$ of the whole, and the water was 2 gallons less than $\frac{1}{9}$ of the whole; how much did the cask hold?
Ans. 54 gallons.

35. A gentleman having \$153 in his purse spent a certain portion, when he found that $\frac{3}{4}$ of what he had spent was equal to $\frac{2}{3}$ of what he had left; how much did he spend?
Ans. \$72.

36. A man bought 15 sheep and 10 hogs for \$95, and he gave \$2 apiece more for the hogs than for the sheep; what was the price of each apiece? *Ans.* Sheep, \$3; hogs, \$5.

37. Two men began business with equal sums of money. One of them gained 216 dollars, and the other lost the same amount, and then one had twice as many dollars as the other; with what sum did each begin? *Ans.* 648 dolls.

38. A confectioner bought 24 barrels of apples and 8 boxes of oranges. A box of oranges cost \$3 more than a barrel of apples, and the whole cost of the apples was double that of the oranges; what was the cost of each?
Ans. Apples, \$6; oranges, \$9.

39. A railroad-train running 24 miles an hour had been despatched 4 hours when another train was sent after it, running at the rate of 32 miles an hour; in how many hours will the second train overtake the first? *Ans.* 12 hours.

40. Five hundred and thirty-four votes were cast at an election, and if the successful candidate had received 24 more votes, he would have doubled his opponent; how many votes did each receive?
Ans. Suc. can., 348; unsuc. can., 186.

41. A grocer bought 90 oranges, 45 cocoanuts and 30 pineapples, paying 4 cents more apiece for the pineapples, and 4 cents less apiece for the oranges, than for the cocoa-

nuts. The entire cost of the oranges, cocoanuts and pineapples was exactly the same; what did he pay apiece for each kind of fruit?

Ans. Oranges, 4 cents; cocoanuts, 8 cents; pineapples, 12 cents.

NOTE.—Let x = the price of a cocoanut.

42. A man bought a horse and buggy, paying \$27 more for the buggy than for the horse, and $\frac{5}{6}$ of the cost of the horse was equal to $\frac{5}{7}$ of the cost of the buggy; what was the cost of each? *Ans.* Horse, \$162; buggy, \$189.

43. After a battle the commander of an army found that $\frac{1}{20}$ of his whole force +200 men were killed, $\frac{1}{10}$ of his army +300 men were wounded, $\frac{1}{5}$ of his force +400 men were taken prisoners, and $\frac{1}{2}$ his army +600 men were left fit for duty; of how many men did the army consist?

Ans. 10,000 men.

44. A boy's age is just $\frac{3}{10}$ of his father's age, but in 12 years the age of the boy will be half that of his father; what is the age of each?

Ans. Boy, 9 years; father, 30 years.

45. A and B began business with equal sums of money. The first year A gained a sum equal to $\frac{1}{3}$ of his original stock, and B lost $\frac{1}{4}$ of his stock. The second year A lost 72 dollars, and B gained 36 dollars, and now B's stock was only $\frac{3}{4}$ of A's stock; what was the original stock of each?

Ans. 360 dollars.

46. Five men, A, B, C, D and E, gained a sum of money, of which B received \$20 less than A; C, \$32 more than B; D, \$10 less than C; and E, \$30 more than D; and the shares of D and E together were equal to the sum of the shares of the other three. What was the share of each?

Ans. A, \$42; B, \$22; C, \$54; D, \$44; E, \$74.

SECTION X.

SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

160. In the examples of the solution of problems which have been given in the preceding Section each problem has involved only *one* unknown quantity, or if two or more have entered into the same question we have been able to express them by means of the same letter. But cases frequently occur in which *several* unknown quantities enter into a calculation.

Suppose we have the equations

$$2x+3y=13,$$
$$3x+2y=12.$$

Here we have *two* equations, each equation containing *two unknown* quantities. The *translation* of this problem into *common* language might be as follows:

"It is required to find two numbers, such that twice the *first*, increased by 3 times the *second*, may be equal to 13, and 3 times the *first*, increased by twice the *second*, may be 12."

Now, let us find an expression for the value of x in each of these equations; in the first,

By transposition, we have $2x=13-3y$;

hence, $x=\frac{13-3y}{2}$.

By transposition in the second, $3x=12-2y$;

hence, $x=\frac{12-2y}{3}$.

We now have an *expression for the value* of x in *each* *equation*, and one of these values of x must be equal to the other; upon the principle that

Quantities that are equal to the same quantity are equal to each other.

Hence, $\frac{13-3y}{2}=\frac{12-2y}{3}$.

The student will notice that we now have an equation containing only *one* unknown quantity, which is to be solved exactly in the same manner that we have illustrated in the preceding Section.

Clearing of fractions, we have

$$39-9y=24-4y;$$

transposing, $-9y+4y=24-39;$

uniting, $-5y=-15;$

changing signs, $5y=15,$

$$y=3.$$

Now, let us substitute this value of y wherever y occurs in one of the original equations; for example, in the first, and we have $2x+9=13,$

$$2x=13-9,$$

$$2x=4,$$

$$x=2.$$

To *prove* whether we have obtained the *true* values of x and y, we go back to our original equations, and substitute the *numerical* values of x and y for those unknown quantities; and if *both* sides of each equation become *identical*,* we know the answers obtained to be correct.

Thus, x being equal to 2 and $y=3$, $2x=4$ and $3y=9$, and their sum is 13; thus *verifying* the first equation.

In the second, $3x=6$ and $2y=6$, and their sum is 12, thus verifying the second equation.

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* Equations are said to be *identical* when the two members are the same, or can be reduced to the same form by performing the operations indicated.

Ex. 2. Given $\left\{ \begin{array}{l} 3x+4y=10 \\ 4x-3y=\ 5 \end{array} \right\}$, to find the values of x and y

In the 1st equation, $3x=10-4y$,

$$x=\frac{10-4y}{3};$$

and in the 2d, $4x=5+3y$,

$$x=\frac{5+3y}{4}.$$

Now, making the two expressions for the value of x equal to each other, we have $\frac{10-4y}{3}=\frac{5+3y}{4}$.

Clearing of fractions, $40-16y=15+9y$;

transposing, $-16y-9y=-40+15$;

uniting, $-25y=-25\cdot$

changing signs, $25y=25$,

$$y=1.$$

Now, substituting in the first equation this value of y for y itself, we have

$$3x+4=10,$$
$$3x=10-4,$$
$$3x=6,$$
$$x=2.$$

The student will notice that the object is, from the *two* equations containing *two* unknown quantities, to deduce *one* equation containing only *one* unknown quantity. This process is called ELIMINATION.* One of the quantities is made to *disappear*, and the method we have used is called

ELIMINATION BY COMPARISON,

because we *compare* the *two* expressions with each other for the value of one of the unknown quantities.

The *translation* of this last example would be about as follows: "Find two numbers, such that 3 times the first, increased by 4 times the second, may be equal to 10; and 4 times the first, diminished by 3 times the second, may be equal to 5."

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* *Elimination*, from the Latin *elimino*, which means *to put out of doors*, *to cause to disappear.*

NOTE.—*To Teachers.* We would again repeat that it will be an exceedingly valuable exercise to require the student to translate into *common* language each problem given in *algebraic* language.

3. Given $\left\{\begin{array}{l} 3x-2y=2 \\ 2x+5y=14 \end{array}\right\}$, to find x and y.

From the first equation, $3x=2+2y$

$$x=\frac{2+2y}{3}.$$

From the second, $2x=14-5y$,

$$x=\frac{14-5y}{2}.$$

Putting the two values of x, equal to each other, we have

$$\frac{2+2y}{3}=\frac{14-5y}{2},$$

$$4+4y=42-15y,$$

$$4y+15y=42-4,$$

$$19y=38,$$

$$y=2.$$

Now, substituting the value of y in the first equation, we have

$$3x=2+4,$$

$$3x=6,$$

$$x=2.$$

After having found the *numerical* value of one of the unknown quantities, we can substitute this value in whichever of the original equations we prefer.

From the foregoing explanations we deduce the following

RULE.—*Find an expression for the value of the* SAME *unknown quantity in each of the equations, and form a new equation by making one of these values equal to the other.*

EXAMPLES.

4. Given $\left\{\begin{array}{l} 3x-2y=8 \\ 4x+\ y=18 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=4, \\ y=2. \end{array}\right.$

5. Given $\left\{\begin{array}{l} 4x+3y=11 \\ 3x-2y=4 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=2, \\ y=1. \end{array}\right.$

6. Given $\left\{\begin{array}{c} x+2y=9 \\ 2x-3y=4 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=5, \\ y=2. \end{array}\right.$

7. Given $\left\{\begin{array}{c} 5x-3y=9 \\ 3x+2y=13 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=3, \\ y=2. \end{array}\right.$

8. Given $\left\{\begin{array}{c} x+5y=11 \\ 4x-3y=21 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=6, \\ y=1. \end{array}\right.$

9. Given $\left\{\begin{array}{c} 7x-5y=13 \\ 3x+3y=21 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=4, \\ y=3. \end{array}\right.$

10. Given $\left\{\begin{array}{c} 4x+3y=26 \\ 3x+2y=19 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=5, \\ y=2. \end{array}\right.$

11. Given $\left\{\begin{array}{c} 5x+4y=39 \\ 2x+3y=24 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=3, \\ y=6. \end{array}\right.$

12. Given $\left\{\begin{array}{c} 8x-7y=5 \\ 5x+2y=35 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=5, \\ y=5. \end{array}\right.$

13. Given $\left\{\begin{array}{c} 7x-2y=26 \\ \frac{x}{3}+\frac{y}{2}=6 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=6, \\ y=8. \end{array}\right.$

14. Given $\left\{\begin{array}{c} 2x+\frac{y}{3}=20 \\ 3x+2y=39 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=9, \\ y=6. \end{array}\right.$

15. It is required to find two such numbers that twice the first, added to 3 times the second, shall be 41; and 6 times the first, diminished by 4 times the second, shall be 6; what are the numbers? *Ans.* First, 7; second, 9.

16. A merchant sold at one time 5 yards of cassimere and 7 yards of silk, and for the whole received \$29; at another time he sold, at the same prices as before, 7 yards of cassimere and 5 yards of silk, and received \$31; what was the price of each per yard? *Ans.* Cassimere, \$3; silk, \$2.

17. Henry bought at one time 7 oranges and 10 apples, and for the whole paid 93 cents; at another time he bought 6 oranges and 12 apples at the same prices as before, and paid 90 cents; what was the cost of the apples and oranges apiece? *Ans.* Oranges, 9 cents; apples, 3 cents.

18. A and B have each a certain sum of money. If 3 times A's money be added to 4 times B's, the sum will be \$85; but if 3 times B's money be subtracted from 4 times A's, the remainder will be \$30; how much has each?

Ans. A, \$15; B, \$10.

19. C's age added to $\frac{2}{3}$ of D's age will give 98 years, but $\frac{2}{3}$ of C's age added to D's will give 92 years; what are their ages?

Ans. C, 66; D, 48.

ELIMINATION BY SUBSTITUTION.

161. Suppose we have the two equations,

$$x-y=3$$
$$x+y=9.$$

Analysis. From the first equation, by transposition, we have

$$x=3+y.$$

Substituting this value of x in the second equation, we have

$$3+y+y=9,$$
$$y+y=9-3,$$
$$2y=6,$$
$$y=3.$$

Now, substituting this value of y for y in one of the original equations—as the first, for example—we have

$$x-3=3,$$
$$x=6.$$

Ans. $x=6$, and $y=3$.

Ex. 2. Given $\left\{\begin{matrix} 2x+3y=7 \\ 3x+2y=8 \end{matrix}\right\}$, to find x and y.

Analysis. From the first equation,

$$2x=7-3y,$$
$$x=\frac{7-3y}{2}.$$

Substituting this value of x in the second equation, we have

$$\frac{21-9y}{2}+2y=8;$$

clearing of fractions, $\quad 21-9y+4y=16,$

$$-9y+4y=16-21,$$
$$-5y=-5;$$

dividing by -1, $\quad 5y=5,$

$$y=1.$$

Now, substituting this value of y in the second equation, we have

$$3x+2=8,$$
$$3x=8-2,$$
$$3x=6,$$
$$x=2.$$

Ans. $x=2$, and $y=1$.

From the foregoing illustrations we derive the following

RULE.—*Find an expression for the value of one of the unknown quantities in one of the equations, and substitute this value for the same unknown quantity in the other equation; we shall thus have an equation containing only one unknown quantity, the value of which can be readily obtained by methods already explained.*

EXAMPLES.

3. Given $\left\{\begin{array}{l} 3x+2y=12 \\ 2x+\ \ y=7 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=2, \\ y=3. \end{array}\right.$

NOTE.—Let the student *translate* each example as he works it.

4. Given $\left\{\begin{array}{l} 4x-3y=13 \\ 2x+2y=10 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=4, \\ y=1. \end{array}\right.$

5. Given $\left\{\begin{array}{l} 5x-2y=5 \\ 2x+3y=21 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=3, \\ y=5. \end{array}\right.$

6. Given $\left\{\begin{array}{l} 2x+5y=15 \\ 5x-3y=22 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=5, \\ y=1. \end{array}\right.$

7. Given $\left\{\begin{array}{l} 6x-4y=2 \\ 2x+3y=18 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=3, \\ y=4. \end{array}\right.$

8. Given $\left\{\begin{array}{l} 4x+3y=23 \\ 6x-2y=2 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=2, \\ y=5. \end{array}\right.$

9. Given $\left\{\begin{array}{l} 3x+\frac{y}{2}=11 \\ 2x+\frac{y}{5}=6 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=2, \\ y=10. \end{array}\right.$

10. Given $\left\{\begin{array}{l} 7x+2y=26 \\ 4x+3y=26 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=2, \\ y=6. \end{array}\right.$

11. Given $\left\{\begin{matrix} \frac{x}{2}+\frac{y}{3}=7 \\ x+y=18 \end{matrix}\right\}$, to find x and y. *Ans.* $\begin{cases} x=6, \\ y=12. \end{cases}$

12. Given $\left\{\begin{matrix} \frac{x}{2}+\frac{y}{4}=10 \\ x+2y=50 \end{matrix}\right\}$, to find x and y. *Ans.* $\begin{cases} x=10, \\ y=20. \end{cases}$

13 Charles and Henry together have 45 marbles, and if twice Henry's marbles be subtracted from 3 times Charles's marbles, the remainder will be 35; how many has each?
Ans. Charles, 25; Henry, 20.

14. James bought at one time 10 oranges and 6 bananas, paying 90 cents for the whole; at another time he bought 5 oranges and 10 bananas, for which he gave 80 cents, paying the same prices apiece as before; what did he pay apiece for them? *Ans.* Oranges, 6 cents; bananas, 5 cents.

15. John and William together have 100 walnuts, and if John had only 20 more he would have twice as many as William; how many walnuts has each?
Ans. John, 60; William, 40.

16. A gentleman bought at one time 5 yards of silk and 3 yards of velvet, for which he gave $25; at another time he bought 3 yards of silk and 5 yards of velvet for $31, at the same rate per yard as before; what was the price of each per yard? *Ans.* Silk, $2; velvet, $5.

17. A father being asked his own age and that of his son, replied, "If ¼ of my age be added to my son's age, the sum will be 28 years; but if my age be added to ¼ of my son's, the result will give 52 years;" what was the age of each?
Ans. Father, 48; son, 16.

18. Find two numbers, such that the sum of six times the first and three times the second may be 36, and the difference between four times the first and five times the second may be 10. *Ans.* 5 and 2.

19. There are two numbers, such that if $\frac{1}{3}$ of the greater be added to $\frac{1}{4}$ of the less, the sum will be 18; and if $\frac{1}{6}$ of the less be taken from $\frac{1}{4}$ of the greater, the remainder will be 5; what are the numbers? *Ans.* 36 and 24.

ELIMINATION BY ADDITION OR SUBTRACTION.

162. Let us take the two equations,

$$4x + 3y = 24,$$
$$5x - 3y = 3.$$

If we add these two equations, member by member, we see that $+3y$ in the first and $-3y$ in the second will *cancel* each other, and we shall have

$$9x = 27,$$
$$x = 3;$$

and substituting this value of x in either of the original equations—the first, for instance—we have

$$12 + 3y = 24,$$
$$3y = 24 - 12,$$
$$3y = 12,$$
$$y = 4.$$

Here we *eliminated* one of the unknown quantities by *addition.*

Ex. 2. Given $\left\{\begin{array}{l} 5x + 4y = 22 \\ 3x + 4y = 18 \end{array}\right\}$, to find x and y.

By *subtracting* the second equation from the first, we find that $+4y$ in the one *cancels* $+4y$ in the other, and we have

$$2x = 4,$$
$$x = 2.$$

Now, substituting this value of x in the second equation, we have

$$6 + 4y = 18,$$
$$4y = 18 - 6,$$
$$4y = 12,$$
$$y = 3.$$

In this example we *eliminated* one of the unknown quantities by *subtraction.*

In the two examples just given it has happened that the *coefficients* of one of the unknown quantities was the *same* in each equation. In most cases, however, they are not the same, but they can be made so by methods now to be explained.

3. Given $\begin{cases} 3x+4y=24 \\ 5x-3y=11 \end{cases}$, to find x and y.

Analysis. Suppose we wish to eliminate y. We see that the coefficients of y in the two equations are different, but we can make them the same if we multiply the first equation by 3 and the second equation by 4; we shall then have

$$9x+12y=72,$$
$$20x-12y=44.$$

Now, *adding* these two equations, we find that $+12y$ *cancels* $-12y$, and we have

$$29x=116,$$
$$x=4.$$

Substituting this value of x in the first equation, we have

$$12+4y=24,$$
$$4y=12,$$
$$y=3.$$

It will be noticed that we multiplied each equation by the coefficient of y in the other equation.

From the foregoing illustrations we derive the following

RULE.—*Multiply or divide the equations in such a way that the coefficients of one of the unknown quantities may be the* SAME *in both equations; then subtract if the signs of the terms to be eliminated are alike, or add if they are unlike.*

NOTE I.—The object of multiplying or dividing is to make the coefficients the same.

NOTE II.—Frequently, the multiplication or division of only *one* of the equations will be sufficient to make the coefficients of one of the unknown quantities the same.

NOTE III.—It is immaterial which quantity is first eliminated.

EXAMPLES.

4. Given $\left\{\begin{array}{l} 4x+3y=25 \\ 3x-2y=6 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=4, \\ y=3. \end{array}\right.$

5. Given $\left\{\begin{array}{l} 5x+4y=14 \\ 4x+3y=11 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=2, \\ y=1. \end{array}\right.$

6. Given $\left\{\begin{array}{l} 4x-5y=4 \\ 2x+3y=24 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=6, \\ y=4. \end{array}\right.$

7. Given $\left\{\begin{array}{l} 5x+2y=15 \\ 6x+3y=21 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=1, \\ y=5. \end{array}\right.$

8. Given $\left\{\begin{array}{l} 6x+5y=43 \\ 7x-3y=6 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=3, \\ y=5. \end{array}\right.$

9. Given $\left\{\begin{array}{l} 3x-2y=9 \\ 4x+3y=29 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=5, \\ y=3. \end{array}\right.$

10. Given $\left\{\begin{array}{l} 5x+4y=38 \\ 2x+3y=18 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=6, \\ y=2. \end{array}\right.$

11. Given $\left\{\begin{array}{l} 7x-5y=3 \\ 3x+4y=32 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=4, \\ y=5. \end{array}\right.$

12. Given $\left\{\begin{array}{l} 6x+4y=34 \\ 4x+3y=23 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=5, \\ y=1. \end{array}\right.$

13. Given $\left\{\begin{array}{l} 2x-7y=2 \\ 5x+6y=52 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=8, \\ y=2. \end{array}\right.$

14. Given $\left\{\begin{array}{l} 8x+3y=69 \\ 6x+5y=71 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=6, \\ y=7. \end{array}\right.$

15. Given $\left\{\begin{array}{l} 4x+\dfrac{y}{2}=43 \\ 5x+\dfrac{y}{3}=52 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=10, \\ y=6. \end{array}\right.$

16. Given $\left\{\begin{array}{l} \dfrac{x}{2}+\dfrac{y}{3}=11 \\ \dfrac{x}{4}+\dfrac{y}{5}=6 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=12, \\ y=15. \end{array}\right.$

17. The ages of John and William are such that if 3 times John's age be added to twice William's, the sum will be 66; but if 3 times William's age be subtracted from 4 times John's age, the remainder will be 20; what are their ages? *Ans.* John, 14; William, 12.

18. Find two numbers such that $\frac{1}{2}$ of the first added to $\frac{1}{3}$ of the second shall make 22, and $\frac{1}{3}$ of the first diminished by $\frac{1}{5}$ of the second shall be 2. *Ans.* 24 and 30.

19. A boy bought at one time 7 oranges and 6 bananas for 60 cents; at another time he bought 6 oranges and 7 bananas, at the same rates as before, paying 57 cents for the whole; what was the price apiece?

Ans. Oranges, 6 cents; bananas, 3 cents.

20. A man invested 90 cents in pears and peaches, paying 3 cents apiece for the pears and 2 cents for peaches. He afterward bought $\frac{1}{4}$ as many pears and $\frac{1}{5}$ as many peaches, at the same rates as before, and paid 21 cents for them; how many of each did he buy?

Ans. 20 pears; 15 peaches.

Equations containing *two* unknown quantities may be solved by any one of the three methods just explained. In most cases the method by addition or subtraction will be found preferable, as it involves fewer fractions than either of the other methods.

SECTION XI.

SIMPLE EQUATIONS CONTAINING THREE UNKNOWN QUANTITIES.

163. Ex. 1. Given $\left\{\begin{array}{ll} 5x+4y+z=24 & (1) \\ 4x+3y+z=19 & (2) \\ 3x-2y+z=6 & (3) \end{array}\right\}$, to find x, y and z.

NOTE.—For convenience of reference, we will designate the several equations by the numbers (1), (2), (3), etc., placed in parentheses.

It is a mere matter of choice which unknown quantity we will first *eliminate*. Suppose, in this instance, we prefer to eliminate z. If we subtract Equation (2) from Eq. (1), the z's will *cancel*, and we shall have
$$x+y=5. \qquad (4)$$

Again: Let us subtract Eq. (3) from Eq. (2), and the z's will again *cancel*, and we shall have $x+5y=13$. (5)

We now have *two* equations containing *two* unknown quantities, which may be solved in the manner explained in the Section embracing two unknown quantities.

Our two equations are $x+5y=13$, (5)

$x+y=5$. (4)

Now, subtracting Eq. (4) from Eq. (5), we find that the x's *cancel*, and we have $4y=8$, (6)

$y=2$. (7)

Substituting this value of y in Eq. (4), we obtain

$$x+2=5,$$
$$x=3.$$

We now have the values of x and y, and substituting these two values in *any one* of the original equations—as, for instance, in Eq. (1)—we have

$$15+8+z=24,$$

or, $$23+z=24,$$

$z=1$. *Ans.* $x=3$, $y=2$, $z=1$.

The student will notice that the whole operation has amounted to this—viz.: that we first combined Eq. (1) with Eq. (2); and secondly, we combined Eq. (2) with Eq. (3); and in each combination *one* of the unknown quantities was dropped out. Only two equations containing two unknown quantities were left, and these were solved in the manner explained in the Section immediately preceding.

The object aimed at is, from the *three* equations containing *three* unknown quantities to *deduce two equations containing two unknown quantities*, and then from these two equations to deduce one equation containing one unknown quantity.

It will not always happen, however, that the unknown quantities can be eliminated without some previous multiplication or division. Sometimes considerable multiplication is necessary.

2. Given $\left\{\begin{array}{ll} x+2y+3z=22 & (1) \\ 2x+3y+4z=34 & (2) \\ 3x+y\ +3z=28 & (3) \end{array}\right\}$, to find x, y and z.

We can *eliminate* whichever quantity we prefer, but it is generally best to take the quantity with the smallest coefficient. In this case suppose we eliminate x.

Let us first compare Eq. (1) and Eq. (2).
If we multiply Eq. (1) by the number 2, we shall have

$$2x+4y+6z=44. \qquad (4)$$

Now, subtracting Eq. (2) from this last, we see that the two x's cancel, and we have

$$y+2z=10. \qquad (5)$$

Again: Let us compare Eq. (1) and Eq. (3).
If we multiply Eq. (1) by the number 3, we shall have

$$3x+6y+9z=66. \qquad (6)$$

Now, subtracting Eq. (3) from Eq. (6), the three x's cancel, and we have

$$5y+6z=38. \qquad (7)$$

We now have *two* equations containing only *two* unknown quantities—viz.:

$$5y+6z=38, \qquad (7)$$
$$y+2z=10. \qquad (5)$$

Now, let us multiply Eq. (5) by 3, so as to make the coefficients of z the *same;* we then have

$$3y+6z=30; \qquad (8)$$

and subtracting Eq. (8) from Eq. (7), we see that the six z's cancel, and we obtain

$$2y=8, \qquad (9)$$
$$y=4. \qquad (10)$$

Substituting this value of y in Eq. (5), we have

$$4+2z=10, \qquad (11)$$
$$2z=6, \qquad (12)$$
$$z=3. \qquad (13)$$

We now have the values of x and y, and substituting these values in *any one* of the original equations—for instance, in Eq. (1)—we have

$$x+8+9=22, \qquad (14)$$
$$x+17=22,$$
$$x=5. \quad \textit{Ans. } x=5,\ y=4,\ z=3.$$

The student will observe that in order to eliminate one of the unknown quantities it is necessary to make *two comparisons or combinations;* but we have *three* methods of making these *two* combinations. First, we may combine the 1st equation with the 2d; or we may combine the 1st with the 3d; or we may combine the 2d with the 3d. Of these *three* methods we can always take our choice, and the student must exercise his own judgment in the selection.

From the foregoing illustrations we derive the following

RULE.—***Combine the equations in pairs; i. e., combine the first with the second, and the first with the third, or the second with the third, so as to eliminate in each case the* SAME *unknown quantity. There will then be formed two new equations, containing only two unknown quantities, which may be solved in the manner previously explained.***

To *prove* whether the results obtained are correct, it is only necessary to substitute the values of the unknown quantities for the quantities themselves in the original equations, and if the two sides of the equation are identical the values found are correct.

The student must not forget that it will not suffice to substitute the values merely in *one* of the equations; they must be substituted in *all* the given equations; or, in other words, each equation must be verified.

EXAMPLES.

3. Given $\left\{\begin{array}{r} 5x+4y+z=49 \\ 2x+3y+z=27 \\ x+y+z=13 \end{array}\right\}$, to find x, y and z. *Ans.* $\left\{\begin{array}{l} x=6, \\ y=4, \\ z=3. \end{array}\right.$

4. Given $\left\{\begin{array}{l} 3x+2y+z=19 \\ 4x-3y+2z=9 \\ 5x+y-3z=20 \end{array}\right\}$, to find x, y and z. *Ans.* $\left\{\begin{array}{l} x=4, \\ y=3, \\ z=1. \end{array}\right.$

5. Given $\left\{\begin{array}{r} x+2y+3z=18 \\ 2x+3y-4z=4 \\ 3x-4y+5z=22 \end{array}\right\}$, to find x, y and z. *Ans.* $\left\{\begin{array}{l} x=5, \\ y=2, \\ z=3. \end{array}\right.$

6. Given $\left\{\begin{array}{l} 7x-4y+2z=22 \\ 4x+3y-3z=10 \\ 5x-4y+4z=28 \end{array}\right\}$, to find x, y and z. *Ans.* $\left\{\begin{array}{l} x=4, \\ y=5, \\ z=7. \end{array}\right.$

7. Given $\left\{\begin{array}{l} 3x-5y+2z=2 \\ 5x+4y-5z=15 \\ 4x-2y+3z=28 \end{array}\right\}$, to find x, y and z. *Ans.* $\left\{\begin{array}{l} x=5 \\ y=5. \\ z=6. \end{array}\right.$

8. Given $\left\{\begin{array}{l} x+\ y+\ z=22 \\ 2x-3y+4z=38 \\ \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=7 \end{array}\right\}$, to find x, y and z. *Ans.* $\left\{\begin{array}{l} x=4, \\ y=6, \\ z=12. \end{array}\right.$

9. Given $\left\{\begin{array}{ll} x+y=16 & (1) \\ x+z=22 & (2) \\ y+z=18 & (3) \end{array}\right\}$, to find x, y and z.

This example may be solved in the same way as the preceding, but by a little device we can somewhat simplify and shorten the operation.

Let us add the *three* equations together; we shall then have

$$2x+2y+2z=56; \qquad (4)$$

dividing by 2, we have $x+\ y+\ z=28.$ (5)

Now, subtracting Eq. (1) from this last, we see that the x's and y's *cancel*, and we have $z=12.$

Again: Let us subtract Eq. (2) from Eq. (5), and the x's and z's cancel, and we obtain $y=6.$

Again: Let us subtract Eq. (3) from Eq. (5), and the y's and z's cancel, and we have $x=10.$ *Ans.* $x=10, y=6, z=12.$

10. Given $\left\{\begin{array}{l} w+x=22 \\ w+y=20 \\ x+y=18 \end{array}\right\}$, to find w, x and y. *Ans.* $\left\{\begin{array}{l} w=12, \\ x=10, \\ y=8. \end{array}\right.$

11. Given $\left\{\begin{array}{l} x+v=10 \\ x+w=9 \\ v+w=11 \end{array}\right\}$, to find x, w and v. *Ans.* $\left\{\begin{array}{l} x=4, \\ w=5, \\ v=6. \end{array}\right.$

12. Find three numbers such that their sum shall be 30; and 3 times the first, diminished by 4 times the second, and increased by twice the third, shall be 12; and twice the fir increased by 3 times the second, and diminished by 4 t the third, shall be 22. *Ans.* 12, 10 and

13. A man bought at one time 6 oranges, 4 lemons and 2 pineapples, for which he paid 76 cents; at another time he bought 4 oranges, 6 lemons and 4 pineapples, which altogether cost 96 cents; and at another time he bought 7 oranges, 4 lemons and 3 pineapples, which cost 94 cents, at the same rates as before; what did he pay apiece for them?

Ans. Oranges, 6 cts.; lemons, 4 cts.; pineapples, 12 cts.

14. Henry and Richard together have 45 walnuts, Henry and Thomas together have 50, and Richard and Thomas together have 55; how many walnuts has each?

Ans. Henry, 20; Richard, 25; Thomas, 30.

15. A man bought a horse, a saddle and a buggy for $215. Now, if twice the price of the horse be diminished by 5 times the price of the saddle, and increased by twice the price of the buggy, the result will be $325; but if 3 times the price of the horse be added to 6 times that of the saddle, and diminished by twice the price of the buggy, the result will give $90; what was the cost of each?

Ans. Horse, $80; saddle, $15; buggy, $120.

16. Three men, A, B and C, jointly built a schoolhouse, for which they paid $450; $\frac{1}{2}$ A's money, added to $\frac{1}{3}$ of B's and $\frac{1}{4}$ of C's, will make $160; but twice A's, increased by 3 times B's, and diminished by 4 times C's, will make $90; how much did each contribute?

Ans. A, $140; B, $150; C, $160.

164. If we have *four* equations, containing *four unknown* quantities, we can solve them upon the same principles that we have employed in the preceding cases. We can first eliminate the same unknown quantity from each of the *four* equations, and this will give us *three* equations containing *three unknown* quantities; and these we can solve exactly as in the foregoing examples.

To eliminate the same unknown quantity from the *four* equations it will be necessary to make three combinations. We can make these combinations in a variety of ways. We may combine the 1st equation with the 2d; or the 1st with the 3d; or the 1st with the 4th; or the 2d with the 3d; or the 2d with the 4th; or the 3d with the 4th.

The student will observe that we can thus make SIX combinations. But only THREE are necessary; hence, he is allowed his choice, and out of the *six* he can select the *three* that he prefers. Thus we see that the more complicated the problem becomes, the greater are the facilities afforded for its solution.

In a similar manner it is possible to solve problems containing *five* or *six*, or even a greater number of, unknown quantities.

NOTE.—The student must remember always in each combination of the equations to cancel or eliminate the same unknown quantity until it entirely disappears from the new equations formed.

165. If we have the equation $x+y=20$, there may be an unlimited number of values for x and y; for if $x=1$, then $y=19$, or if $x=2$, then $y=18$, and so on. The question is then said to be *indeterminate; i. e.* incapable of a *definite* answer.

But if, in addition to the equation $x+y=20$, we have another equation, $\frac{x}{2}+\frac{y}{3}=8$, it will be found now that x and y have each only *one* value—viz. $x=8$ and $y=12$. The problem now is said to be *determinate*;* *i. e.* it admits of a *definite* answer.

One of the equations serves as a check or limitation on the other; hence, when we have a problem involving *two* unknown quantities, it is necessary to have *two* equations; if *three* unknown quantities enter into the calculation, we must have *three* equations; and, universally, there must be as many equations as there are unknown quantities.

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* *Determinate*, from the Latin *determino, to limit, to bound.*

Furthermore, each of the equations employed must belong to the class known as *independent* equations; *i. e.* equations which express *different* conditions. Thus, in the example given above, where $x+y=20$ and $\frac{x}{2}+\frac{y}{3}=8$, these two equations are *independent*, because they express *different* conditions. But $x+y=20$ and $\frac{x}{2}+\frac{y}{2}=10$ are *dependent* equations, because they express exactly the *same* conditions; for one of these equations may be derived from the other simply by multiplying or dividing.

17. Given $\left\{\begin{array}{l} v+2x-3y+z=6 \\ 2v-x+2y+2z=16 \\ 3v+3x+y-3z=6 \\ 5v-4x+4y+4z=26 \end{array}\right\}$, to find v, x, y and z.

Ans. $v=2$, $x=4$, $y=3$, $z=5$.

18. Given $\left\{\begin{array}{l} 2x+3y-4z+w=13 \\ 4x+2y+3z-2w=9 \\ 5x-4y+2z+3w=24 \\ x+5y-6z+4w=27 \end{array}\right\}$, to find x, y, z and w.

Ans. $x=3$, $y=2$, $z=1$, $w=5$.

SECTION XII.

INVOLUTION.

166. WHEN a quantity is multiplied by itself the product is called a POWER; thus,

$3\times3=9$, the second power, or square of 3.
$3\times3\times3=27$, the third power, or cube of 3.
$3\times3\times3\times3=81$, the fourth power of 3.
$3\times3\times3\times3\times3=243$, the fifth power of 3, etc.

Again: $a \times a = aa$, the second power or square of a.
$a \times a \times a = aaa$, the third power, or cube of a.
$a \times a \times a \times a = aaaa$, the fourth power of a, etc.

167. A POWER is the product arising from multiplying a quantity into itself a certain number of times. The quantity itself is called the *first* power. It is also called a ROOT of the other powers; thus, 9 is the second *power* of 3, and 3 is the second, or the square *root*, of 9. So 27 is the third *power* of 3, and 3 is the third, or the cube *root*, of 27, and so on.

168. The process of raising a quantity to any required power is called INVOLUTION.

169. Instead of actually performing the multiplication, we may *indicate* the power by placing a small figure, called an *exponent* or *index*, at the right of the quantity and a little above it (23).

Thus, $a \times a = aa$, is written a^2, and is read, "The second power of a, or the square of a;"
$a \times a \times a = aaa$ is written a^3, and is read, "The third power of a, or the cube of a;"
$a \times a \times a \times a = aaaa$ is written a^4, and is read, "The fourth power of a;" and so on.

Here it will be seen that the index of the second power is 2; of the third power, 3; of the fourth power, 4; and so on. The index of the first power is 1, but this is seldom written.

Ex. 1. What is the square of $5ax$?

SOLUTION. $(5ax)^2 = 5ax \times 5ax = 5 \times 5 \times a \times a \times x \times x = 25a^2x^2$.

2. What is the cube of $4bmy$?

SOLUTION. $(4bmy)^3 = 4bmy \times 4bmy \times 4bmy = 64b^3m^3y^3$.

3. What is the fourth power of $3aby$?

SOLUTION. $(3aby)^4 = 3aby \times 3aby \times 3aby \times 3aby = 81a^4b^4y^4$.

Hence, to raise a quantity to any power we have the following

RULE.—*Multiply the quantity into itself until it is taken as a factor as many times as there are units in the index of the required power.*

NOTE.—In raising a quantity to any power the number of multiplications is always *one* less than the number of units in the index of the power; thus, *one* multiplication gives the square, or second power; *two* multiplications, the cube, or third power; *three* multiplications, the fourth power; and so on.

EXAMPLES.

4. What is the square of $7abc$? *Ans.* $49a^2b^2c^2$.
5. What is the cube of $5axy$? *Ans.* $125a^3x^3y^3$.
6. What is the fourth power of $4bcy$? *Ans.* $256b^4c^4y^4$.
7. What is the fifth power of $5mpy$? *Ans.* $3125m^5p^5y^5$.
8. What is the sixth power of $3aby$? *Ans.* $729a^6b^6y^6$.
9. What is the third power of $9xyz$? *Ans.* $729x^3y^3z^3$.
10. What is the third power of $15abc$? *Ans.* $3375a^3b^3c^3$.
11. What is the square of $25abmy$? *Ans.* $625a^2b^2m^2y^2$.
12. What is the fifth power of $7bmx$? *Ans.* $16807b^5m^5x^5$.
13. What is the sixth power of $2abyz$? *Ans.* $64a^6b^6y^6z^6$.

170. Sometimes the quantity to be involved is already a power.

Ex. 1. What is the third power of a^2?

SOLUTION. $(a^2)^3 = a^2 \times a^2 \times a^2 = a^{2\times 3} = a^6$.

2. What is the fourth power of a^3?

SOLUTION. $(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3 = a^{3\times 4} = a^{12}$.

3. What is the fifth power of a^2?

SOLUTION. $(a^2)^5 = a^2 \times a^2 \times a^2 \times a^2 \times a^2 = a^{2\times 5} = a^{10}$.

171. From the foregoing illustrations we see that

A quantity which is already a power is involved by mul-

tiplying its index into the index of the power to which it is to be raised.

EXAMPLES.

4. What is the third power of y^4? *Ans.* y^{12}.

5. What is the fourth power of a^2b^3? *Ans.* ——.

6. What is the third power of $x^2y^3z^4$? *Ans.* $x^6y^9z^{12}$.

7. What is the fifth power of ab^2c^3? *Ans.* ——.

8. What is the sixth power of $a^2b^3cy^4$? *Ans.* $a^{12}b^{18}c^6y^{24}$.

9. What is the fourth power of $a^2b^3x^4$? *Ans.* ——.

10. What is the third power of $5a^2b^3$? *Ans.* $125a^6b^9$.

In examples like the last it is necessary to combine (**169**) with (**171**); *i. e.* we must actually involve the coefficient by multiplying it into itself the required number of times, and multiply the index of the letter or letters by the index of the required power.

11. What is the fourth power of $5ab^2c^3$? *Ans.* $625a^4b^8c^{12}$.

12. What is the fifth power of $3a^2b^4y$? *Ans.* ——.

13. What is the third power of $8a^3b^4y^2$? *Ans.* $512a^9b^{12}y^6$.

14. What is the fourth power of $6a^2cy^3b^4$? *Ans.* ——.

15. What is the third power of $10a^4b^3y^2$? *Ans.* $1000a^{12}b^9y^6$.

16. What is the sixth power of $3a^2bc^5$? *Ans.* ——.

17. What is the fifth power of $6a^3b^4c^6$? *Ans.* $7776a^{15}b^{20}c^{30}$.

18. What is the fourth power of $7a^2x^3y^4$? *Ans.* ——.

172. When quantities are involved attention must be paid to the *sign* to be prefixed to the product:

$(-a)^2 = -a \times -a = +a^2$, the second power of $(-a)$.

$(-a)^3 = -a \times -a \times -a = -a^3$, the third power of $(-a)$.

$(-a)^4 = -a \times -a \times -a \times -a = +a^4$, the fourth power of $(-a)$.

$(-a)^5 = -a \times -a \times -a \times -a \times -a = -a^5$, the fifth power of $(-a)$.

No proof is required to show that when the root is *posi-*

tive all its powers are positive, because a positive quantity multiplied into itself any number of times will necessarily give a positive product. Hence, we deduce the general principle:

If the root is positive, ALL *its powers are positive; but if the root is negative, the* ODD *powers are negative, while the* EVEN *powers are positive.*

19. What is the third power of $-5a^2b^3y$? *Ans.* $-125a^6b^9y^3$.

20. What is the fourth power of $4a^3b^2y^4$? *Ans.* $256a^{12}b^8y^{16}$.

21. What is the fifth power of $-3a^3b^2x$? *Ans.* $-243a^{15}b^{10}x^5$.

22. What is the second power of $-15a^2b^3c^4$? *Ans.* $225a^4b^6c^8$.

23. What is the third power of $7ab^2c^4$? *Ans.* $343a^3b^6c^{12}$.

24. What is the fifth power of $-4a^2by^3$? *Ans.* $-1024a^{10}b^5y^{15}$.

25. What is the square of $\frac{3a}{4b}$?

OPERATION. $\left(\frac{3a}{4b}\right)^2=\frac{3a}{4b}\times\frac{3a}{4b}=\frac{9a^2}{16b^2}$. *Ans.* $\frac{9a^2}{16b^2}$.

26. What is the third power of $\frac{4x}{5y}$?

OPERATION. $\left(\frac{4x}{5y}\right)^3=\frac{4x}{5y}\times\frac{4x}{5y}\times\frac{4x}{5y}=\frac{64x^3}{125y^3}$. *Ans.* $\frac{64x^3}{125y^3}$.

Hence, to raise a fraction to any power,

173. *Involve the numerator and the denominator separately.*

27. What is the square of $\frac{7a^3}{m^2}$? *Ans.* $\frac{49a^6}{m^4}$.

28. What is the third power of $\frac{4a^2b^3}{my}$? *Ans.* ——.

29. What is the fourth power of $\frac{3ab^2}{5a^2x^3}$? *Ans.* $\frac{81a^4b^8}{625a^8x^{12}}$.

30. What is the third power of $\frac{6a^3x^2}{7ay^3}$? *Ans.* ——.

31. What is the second power of $\frac{12ax^2y^3}{13x^2b^3}$?

Ans. $\frac{144a^2x^4y^6}{169x^4b^6}$.

32. What is the third power of $\frac{4x^2y^4}{5ax}$? *Ans.* ——.

33. What is the fourth power of $\frac{-2x^2y^5}{5a^3b^2}$? *Ans.* $\frac{16x^8y^{20}}{625a^{12}b^8}$.

34. What is the second power of $\frac{12ab^2c^3}{15xy}$? *Ans.* ——.

174. To Raise a Polynomial to any Power.

Any power of a polynomial may be found, like that of a monomial, by simply multiplying it into itself the required number of times. Suppose we wish the *fifth* power of $(a+b)$.

OPERATION.

$$\begin{array}{rl}
 & a+b = \text{1st power.} \\
 & \underline{a+b} \\
 & a^2+ab \\
 & \underline{\quad +ab+b^2} \\
(a+b)^2 = & a^2+2ab+b^2 = \text{2d power.} \\
 & \underline{a+b} \\
 & a^3+2a^2b+ab^2 \\
 & \underline{\quad +a^2b+2ab^2+b^3} \\
(a+b)^3 = & a^3+3a^2b+3ab^2+b^3 = \text{3d power.} \\
 & \underline{a+b} \\
 & a^4+3a^3b+3a^2b^2+ab^3 \\
 & \underline{\quad +a^3b+3a^2b^2+3ab^3+b^4} \\
(a+b)^4 = & a^4+4a^3b+6a^2b^2+4ab^3+b^4 = \text{4th power.} \\
 & \underline{a+b} \\
 & a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
 & \underline{\quad +a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5} \\
(a+b)^5 = & a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 = \text{5th power.}
\end{array}$$

In the same manner, were we to involve $(a-b)$ to the fifth power, we should obtain this result—viz.:

$$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

The only difference between raising $(a+b)$ and $(a-b)$ to any power is simply in the *signs*. In the case of $(a+b)$ the signs of *all* the terms are positive. In the case of $(a-b)$ the signs are alternately positive and negative; *i. e.* all the *odd* terms are positive, while all the *even* terms are negative.

EXAMPLES.

2. What is the fourth power of $(x+y)$?
Ans. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

3. Involve $(c-d)$ to the third power.
Ans. $c^3 - 3c^2d + 3cd^2 - d^3$.

4. What is the fifth power of $(m+n)$?
Ans. $m^5 + 5m^4n + 10m^3n^2 + 10m^2n^3 + 5mn^4 + n^5$.

5. What is the square of $(2a+3b)$? *Ans.* $4a^2 + 12ab + 9b^2$.

6. Involve $(2b-5c)$ to the third power.
Ans. $8b^3 - 60b^2c + 150bc^2 - 125c^3$.

7. What is the fourth power of $(3x+2y)$?
Ans. $81x^4 + 216x^3y + 216x^2y^2 + 96xy^3 + 16y^4$.

8. What is the third power of $(3x^2+2y)$?
Ans. $27x^6 + 54x^4y + 36x^2y^2 + 8y^3$.

9. Involve $(a+b-c)$ to the second power.
Ans. $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.

BINOMIAL THEOREM.

175. Any binomial may be raised to any power by multiplying it into itself in the manner we have explained in the preceding Article. But when a high power is required, the operation becomes exceedingly tedious. A very simple and convenient method has been devised, by which any power of a binomial may be obtained without the labor of actual multiplication. This method is known as the BINOMIAL THEOREM. It was discovered by Sir Isaac Newton in 1666, and is one of the most ingenious and labor-saving discoveries ever made in mathematics.

If we take the binomial $(a+x)$, and form a series of powers by actual multiplication, we shall obtain the following results:

$$\begin{aligned}
&(a+x)^1=a+x,\\
&(a+x)^2=a^2+2ax+x^2,\\
&(a+x)^3=a^3+3a^2x+3ax^2+x^3,\\
&(a+x)^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4,\\
&(a+x)^5=a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5,\\
&(a+x)^6=a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6.
\end{aligned}$$

The several powers of $(a-x)$ are exactly the same as those of $(a+x)$, except that in $(a-x)$ the signs of all the *odd* terms are +, and the *even* terms are −.

By examining this series of powers we shall discover several important particulars.

I. We find that the *exponents* preserve an *invariable* order throughout the whole.

In the square the exponents of $\begin{cases} a \text{ are } 2, 1, 0; \\ x \text{ are } 0, 1, 2. \end{cases}$

In the cube the exponents of $\begin{cases} a \text{ are } 3, 2, 1, 0; \\ x \text{ are } 0, 1, 2, 3. \end{cases}$

In the fourth power the exponents of $\begin{cases} a \text{ are } 4, 3, 2, 1, 0; \\ x \text{ are } 0, 1, 2, 3, 4. \end{cases}$

In the fifth power the exponents of $\begin{cases} a \text{ are } 5, 4, 3, 2, 1, 0; \\ x \text{ are } 0, 1, 2, 3, 4, 5. \end{cases}$

In the sixth power the exponents of $\begin{cases} a \text{ are } 6, 5, 4, 3, 2, 1, 0; \\ x \text{ are } 0, 1, 2, 3, 4, 5, 6. \end{cases}$

II. We observe that the exponents of a in the *first* term and of x in the *last* term are each equal to the index of the power. Thus, in the expansion of $(a+x)^5$ the exponent of a in the *first* term is 5, and the exponent of x in the *last* term is 5 also.

III. We also notice that the exponents of a regularly *decrease* by 1 from the beginning, and that the exponents of x regularly *increase* by 1; furthermore, that the *sum* of the exponents in each term is always the same, and always equal to the index of the power.

Thus, in $(a+x)^5$ the *sum* of the exponents of a and x in each term is invariably equal to 5.

In $(a+x)^6$ the *sum* of the exponents in each term is always 6.

IV. We also observe that x *begins* in the *second* term with the exponent 1, and uniformly *increases* by 1.

V. The number of terms is always *one more* than the exponent of the power required; thus,

In the square there are three terms;
In the cube there are four terms;
In the fourth power there are five terms;
In the fifth power there are six terms; and so on.

By remembering these principles we shall find no difficulty in writing out the *exponents* of any power.

Suppose we wish the exponents of $(a+x)^6$. By applying what we have said, we shall have

$$a^6+a^5x+a^4x^2+a^3x^3+a^2x^4+ax^5+x^6.$$

176. We will now examine the *coefficients.*

By inspecting the series of powers in **175** we find that the coefficient of the *first* term is *always* 1, and that the coefficient of the *second* term is always the *same* as the index of the power. Thus, in $(a+x)^6$ the coefficient of the first term is 1; the coefficient of the second term is 6, the *same* as the index of the power. To get the coefficient of the third term, we multiply 6 (the coefficient of the second term) by 5, the index of (a) in that term, and divide the product by 2, which is the index of x increased by 1, giving 15 for the third coefficient. For the fourth coefficient we multiply 15 (the third coefficient) by 4, the index of (a) in the third term, and divide the product by 3, the index of x increased by 1, which gives 20 as the fourth coefficient. For the fifth coefficient we multiply 20 (the fourth coefficient) by 3, the index of (a) in the fourth term, and divide the product by 4, the index of x increased by 1, which gives 15 as the fifth coefficient. To get the sixth coefficient we multiply 15 (the fifth coefficient) by 2, the index of (a) in the fifth term, and divide the product by 5, the index of x increased by 1, giving 6 as the sixth coefficient. For the seventh coefficient we multiply 6 (the sixth coefficient) by 1, the index of (a) in the sixth term, and divide the product by 6, the index of x increased by 1, giving 1 as the seventh and *last* coefficient.

The OPERATION would be as follows:

First term.	Second.	Third.	Fourth.	Fifth.	Sixth.	Seventh.
1,	6,	$\frac{6\times5}{2}$,	$\frac{15\times4}{3}$,	$\frac{20\times3}{4}$,	$\frac{15\times2}{5}$,	$\frac{6\times1}{6}$.
Or, 1,	6,	15,	20,	15,	6,	1.

Now, prefixing these coefficients to their respective terms, we have

$$(a+x)^6=a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6.$$

2. Involve $(a+x)$ to the seventh power.

Writing the several terms without the coefficients, we have

$$a^7+a^6x+a^5x^2+a^4x^3+a^3x^4+a^2x^5+ax^6+x^7.$$

The OPERATION for obtaining the coefficients would be as follows:

First term.	Second.	Third.	Fourth.	Fifth.	Sixth.	Seventh.	Eighth.
1,	7,	$\frac{7\times6}{2}$,	$\frac{21\times5}{3}$,	$\frac{35\times4}{4}$,	$\frac{35\times3}{5}$,	$\frac{21\times2}{6}$,	$\frac{7\times1}{7}$.
Or, 1,	7,	21,	35,	35,	21,	7,	1.

Prefixing these coefficients to their respective terms, we have

$$(a+x)^7=a^7+7a^6x+21a^5x^2+35a^4x^3+35a^3x^4+21a^2x^5+7ax^6+x^7.$$

The student cannot fail to notice the beautiful symmetry in the coefficients. The coefficients of the *first* and the *last* terms are always 1; the coefficients of the *second* and the *next* to the last are always the same; and, universally, the coefficients of terms equally distant from the first and the last are always equal.

In consequence of this property it will only be necessary to find the coefficients of the first half of the terms, and the remaining half will be the same as the first, only written in an inverted order.

NOTE.—In a binomial the first term is called the *leading* quantity, and the second term the *following* quantity. Thus, in $(a+x)$, (a) is the *leading*, and x the *following*, quantity.

177. All that we have said may be summed up in the following Rule, which is, substantially, Sir Isaac Newton's Theorem:

RULE.—*In any power of a binomial the index of the leading quantity begins in the first term with the index of the power, and regularly decreases by 1. The index of the following quantity commences with 1 in the second term, and continually increases by 1.*

The coefficient of the first term is 1; that of the second is the index of the power; and, if the coefficient of any term be multiplied by the index of the leading quantity in that term, and divided by the index of the following quantity increased by 1, it will give the coefficient of the succeeding term.

EXAMPLES.

3. Involve $(a-x)$ to the third power.

Ans. $a^3-3a^2x+3ax^2-x^3$.

4. What is the fourth power of $(m+x)$?

Ans. $m^4+4m^3x+6m^2x^2+4mx^3+x^4$.

5. Involve $(x+y)$ to the fifth power.

Ans. $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$.

6. What is the sixth power of $(b+x)$?

Ans. $b^6+6b^5x+15b^4x^2+20b^3x^3+15b^2x^4+6bx^5+x^6$.

7. What is the seventh power of $(p+q)$?

Ans. $p^7+7p^6q+21p^5q^2+35p^4q^3+35p^3q^4+21p^2q^5+7pq^6+q^7$.

8. What is the eighth power of $(a+b)$?

Ans. $a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+$
$28a^2b^6+8ab^7+b^8$.

9. What is the ninth power of $(m+x)$?

Ans. $m^9+9m^8x+36m^7x^2+84m^6x^3+126m^5x^4+126m^4x^5+$
$84m^3x^6+36m^2x^7+9mx^8+x^9$.

10. What is the fourth power of $(2a^2+3y)$?

OPERATION.

$$(2a^2+3y)^4=(2a^2)^4+4(2a^2)^3(3y)+6(2a^2)^2(3y)^2+4(2a^2)^1(3y)^3+(3y)^4.$$

Now, actually performing the operations *indicated*, we have

$$(2a^2+3y)^4=16a^8+96a^6y+216a^4y^2+216a^2y^3+81y^4.$$

Or we may employ another method:

Let $2a^2=x$ and $3y=b$; then, $(2a^2+3y)^4=(x+b)^4$.

Now, $(x+b)^4=x^4+4x^3b+6x^2b^2+4xb^3+b^4$.

Now, substituting the values of x and b in this last formula, remembering that

$x = 2a^2$ and $b = 3y$,
$x^2= 4a^4$ " $b^2= 9y^2$,
$x^3= 8a^6$ " $b^3=27y^3$,
$x^4=16a^8$ " $b^4=81y^4$,

we have $(2a^2+3y)^4=16a^8+4\times 8a^6\times 3y+6\times 4a^4\times 9y^2+4\times 2a^2\times 27y^3+81y^4$,

or, $(2a^2+3y)^4=16a^8+96a^6y+216a^4y^2+216a^2y^3+81y^4$.

To beginners in Algebra the method by substitution presents some advantages.

11. What is the third power of $(2a^2+2b^2)$?

Ans. $8a^6+24a^4b^2+24a^2b^4+8b^6$.

12. Involve $(2a^3+4x)$ to the fourth power.

Ans. $16a^{12}+128a^9x+384a^6x^2+512a^3x^3+256x^4$.

13. Involve $(3x-2y)$ to the fifth power.

Ans. $243x^5-810x^4y+1080x^3y^2-720x^2y^3+240xy^4-32y^5$.

14. What is the fourth power of $(3a+2x^2)$?

Ans. $81a^4+216a^3x^2+216a^2x^4+96ax^6+16x^8$.

178. Through the device of substitution any polynomial may be raised to any power by the Binomial Theorem.

Suppose we are required to raise $(a+b+c)$ to the third power.

Let us put $x=(a+b)$; then, $(a+b+c)^3=(x+c)^3$.

But $(x+c)^3=x^3+3x^2c+3xc^2+c^3$.

Now, substituting $(a+b)$ for x in the *expansion*, we have

$$(a+b+c)^3=(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3.$$

Actually performing the operations indicated, we have

$$(a+b+c)^3=a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3.$$

2. What is the third power of $(m+n+x)$?

Ans. $m^3+3m^2n+3mn^2+n^3+3m^2x+6mnx+3n^2x+3mx^2+3nx^2+x^3$.

In a manner entirely similar we could involve a trinomial to the fourth, fifth or any higher power.

Upon the same principle we can raise a polynomial consisting of *more* than three terms to any power. For example, if we were required to raise $(a+b+c+d)$ to the third power, we could put $(a+b)=x$ and $(c+d)=y$; then $(a+b+c+d)^3=(x+y)^3$. Expanding $(x+y)^3$, and restoring the respective values of x and y, and performing the operations indicated, we should have the expansion of $(a+b+c+d)^3$. The same process might be extended to *any* power of any polynomial.

SECTION XIII.

EVOLUTION.

179. EVOLUTION* is the reverse of INVOLUTION. In INVOLUTION the *root* is given to find a *power;* in EVOLUTION the *power* is given to find a *root.*

180. A ROOT of a quantity is a factor which, being multiplied by itself a certain number of times, will produce the quantity.

The SQUARE ROOT of a quantity is that factor which, being multiplied by itself *once,* will produce the quantity; thus, the *square root* of 36 is 6, for $6 \times 6 = 36$; the square root of $9x^2$ is $3x$, for $3x \times 3x = 9x^2$.

181. The CUBE ROOT of a quantity is that factor which, being multiplied by itself *twice,* will produce the quantity; thus, the cube root of 27 is 3, for $3 \times 3 \times 3 = 27$; the cube root of x^6 is x^2, for $x^2 \times x^2 \times x^2 = x^6$.

182. A *perfect square* is a quantity whose *square root* can be *exactly* found. Thus, 1, 4, 9, 25, etc. are *perfect squares;* so are a^2, a^4, a^8, a^{10}, etc. Any quantity whose index can be exactly divided by 2 is a perfect square.

183. A *perfect cube* is a quantity whose *cube root* can be *exactly* found; thus, 1, 8, 27 and 64 are perfect cubes; so are a^3, y^6, b^9 and x^{12}. Any quantity whose index is exactly divisible by 3 is a perfect cube.

If its index is exactly divisible by 4, then it is a *perfect fourth power;* as, for example, a^4, x^8, y^{12}, etc.

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* ***Evolution,*** from the Latin ***evolvo,*** meaning ***to roll out, to unwind.***

184. Those quantities whose roots cannot be exactly found are called *imperfect* powers, and their roots are called *irrational** quantities, or surds†. Thus, the square roots of 5 and 7 are *irrational* or *surd* quantities; so are the cube roots of 12 and 20.

185. A RATIONAL quantity is one whose value can be expressed in finite terms, or without any radical sign or fractional index; thus, 3, $5a$, $\frac{3x}{4}$, etc. are *rational* quantities.

186. A quantity may be a perfect power of one degree and an imperfect power of another degree. Thus, 25 is a perfect square, but an imperfect cube; whilst 27 is a perfect cube, but an imperfect square; but 64 is both a perfect square and a perfect cube.

187. The roots of quantities are indicated by means of the radical sign $\sqrt{}$, with the proper index annexed, or by fractional indices placed at the right hand of the quantity.

Thus, $\sqrt[2]{a}$, or $a^{\frac{1}{2}}$, expresses the square root of a;
$\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, denotes the cube root of a;
$\sqrt[4]{a}$, or $a^{\frac{1}{4}}$, indicates the fourth root of a, etc.

188. The INDEX of the root is the figure placed over the radical sign to indicate what root is required. Thus, the *index* of the cube root is 3, of the fourth root is 4, etc. The index of the square root is not usually written, but the root is indicated by simply writing the radical sign over the quantity; thus, $\sqrt{2}$ denotes the square root of 2; $\sqrt{x}$ indicates the square root of x.

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* *Irrational,* from the Latin *in* and *rationalis,* meaning *not susceptible of ratio; i. e.* impossible to be expressed in the form of a ratio.

† *Surd,* from the Latin *surdus,* signifying *deaf; i. e.* incapable of being heard, because incapable of being exactly expressed in finite terms.

SIGNS OF ROOTS.

189. I. *The odd root of any quantity has the same sign as the quantity.*

For a positive quantity raised to any power is always positive, and a negative quantity raised to an *odd* power is always negative; therefore, the odd root of a positive quantity is *positive*, and the odd root of a negative quantity is *negative;* thus, $\sqrt[3]{64} = +4$; $\sqrt[3]{-27} = -3$.

II. *An even root of a positive quantity is either positive or negative.*

For a positive quantity raised to an even power is positive, and a negative quantity raised to an even power is also positive; therefore, the even root of a positive quantity can be either positive or negative. Thus, the square root of 36 is either $+6$ or -6; for $+6$ multiplied into itself produces $+36$, and -6 multiplied into itself also gives $+36$. An *even* root of a positive quantity is therefore said to be *ambiguous*, and this ambiguity or uncertainty is indicated by prefixing both + and − to the root; thus,

$$\sqrt{64} = \pm 8;\ \sqrt{16} = \pm 4;\ \sqrt{x^2} = \pm x.$$

III. *An* EVEN *root of a negative quantity is impossible.*

Thus, the square root of -49 *cannot* be found, for it must be either $+7$ or -7; but $+7$ multiplied into itself produces $+49$, and -7 multiplied into itself also gives $+49$; hence, there being no quantity that multiplied into itself will give -49, the square root of it cannot be obtained, and is therefore said to be *impossible.* Such quantities are frequently called *imaginary.*

Observation. If such quantities are purely imaginary, the student may be inclined to ask why they are so often introduced into Algebra, and so much labor and time expended on them. The reply to this is, that these *imaginary* quantities have their uses in enabling us to detect some inconsistency or impossibility either in our original statement or in our course of reasoning.

When we are attempting to solve a problem, and meet one of these *imaginary* quantities, this very fact indicates either that our original statement was absurd, or that, in our subsequent reasoning, we have violated some established mathematical principle.

190. To extract the *square* root of any quantity is to find a factor which, multiplied by itself *once*, will produce the quantity.

To extract the cube root of any quantity is to find a factor which, multiplied by itself *twice*, will produce the quantity.

CASE I.

191. To Extract any Root of a Monomial.

As *evolution* is the reverse of *involution*, we can derive the method of extracting the root from observing how we formed the power.

In *squaring* a monomial we raise the numerical coefficient to the *second* power, and multiply the exponent of each of the letters by 2 (**171**). Hence, to extract the *square root* we *reverse* the process; *i. e.* we extract the square root of the numerical coefficients, and divide the exponent of each of the literal factors by 2. Thus, $\sqrt{16a^2x^4y^6}$ is $4ax^2y^3$.

In *cubing* a monomial we raise the numerical coefficient to the *third* power, and multiply the exponent of each of the letters by 3 (**171**). Hence, to extract the *cube root* we reverse the operation; *i. e.* we take the cube root of the numerical coefficients, and divide the exponent of each of the literal factors by 3. In a similar manner we proceed with other roots. Hence, we deduce the following

RULE.—*I. Extract the root of the numerical coefficient, as in Arithmetic.*

II. Divide the exponent of each of the literal factors by the index of the required root.

NOTE.—Remember to prefix the proper sign to the root, in accordance with **189**.

EXAMPLES.

1. What is the square root of [illegible]? Ans. ± [illegible].
2. What is the square root of [illegible]? Ans. ± [illegible].
3. What is the cube root of [illegible]? Ans. [illegible].
4. What is the cube root of − [illegible]? Ans. − [illegible].
5. Find the square root of [illegible]. Ans. ± [illegible].
6. Extract the cube root of [illegible]. Ans. [illegible].
7. Find the cube root of − [illegible]. Ans. − [illegible].
8. What is the fourth root of [illegible]? Ans. ± [illegible].
9. Find the fifth root of 32[illegible]. Ans. 2[illegible].
10. What is the value of $\sqrt{121x^2y^4}$? Ans. $\pm 11xy^2$.
11. What is the value of $\sqrt[3]{216a^3b^6}$? Ans. $6ab^2$.
12. What is the value of $\sqrt[4]{81c^4x^8}$? Ans. $\pm 3cx^2$.
13. Extract the square root of $576a^2b^4x^6$. Ans. $\pm 24ab^2x^3$.
14. Find the cube root of $-125a^3x^6$. Ans. $-5ax^2$.
15. What is the fourth root of $256x^4y^{12}$? Ans. $\pm 4xy^3$.
16. Extract the fifth root of $-243b^5y^{10}$. Ans. $-3by^2$.
17. What is the sixth root of $64c^6x^{12}$? Ans. $\pm 2cx^2$.
18. Find the square root of $625a^{2m}b^{4m}c^2$. Ans. $\pm 25a^mb^{2m}c$.
19. Extract the square root of $1296a^8b^4c^2$. Ans. $\pm 36a^4b^2c$.
20. Find the cube root of $343a^9b^6d^3$. Ans. $7a^3b^2d$.
21. Extract the square root of $2025b^4c^2y^6$. Ans. $\pm 45b^2cy^3$.
22. What is the square root of $\dfrac{4a^2x^2}{9b^4y^6}$?

In raising a fraction to any power we involve the numerator and the denominator separately; hence, to get the root we reverse the process, and extract the root of the numerator and the denominator separately; hence, the square root of $\dfrac{4a^2x^2}{9a^4y^6}$ is $\dfrac{\pm 2ax}{\pm 3a^2y^3}$.

23. What is the square root of $\dfrac{16a^4b^6}{49x^8y^2}$? Ans. $\dfrac{\pm 4a^2b^3}{\pm 7x^4y}$.

24. What is the cube root of $\frac{8a^3b^6}{27x^9y^3}$? Ans. $\frac{2ab^2}{3x^3y}$.

25. Find the square root of $\frac{25a^2b^6}{81x^4y^8}$. Ans. $\frac{\pm 5ab^3}{\pm 9x^2y^4}$.

CASE II.

192. To Extract the Square Root of a Polynomial.[*]

If we analyze the process of squaring a polynomial, we shall discover a regular law which will enable us to extract the square root of any quantity whatever.

Now, by **174** we know that the square of $(b+x)$ is $b^2+2bx+x^2$; *i. e.* it is equal to the *square of the first term*, + *twice the product of the first by the second*, + *the square of the second.*

Now let us extract the *square root* of $b^2+2bx+x^2$.

OPERATION.

$$
\begin{array}{l}
b^2+2bx+x^2(b+x \\
\underline{b^2} \\
2b+x)2bx+x^2 \\
\underline{2bx+x^2}
\end{array}
$$

Analysis. To extract the *square root* will be to *reverse* the operation we have just performed.

The *square root* of b^2 is b, which we write in the quotient as the first term of the root; subtracting the square of b from the polynomial, we find the remainder to be $2bx+x^2$. This remainder, $2bx+x^2$, is equal to *twice the first term multiplied by the second*, + *the square of the second.*

We already have the first term, b; therefore, dividing $2bx$ by $2b$ (*i. e.* by *twice the first term*), we obtain x, the *second* term of the root; and this, annexed to $2b$, gives the *complete divisor*, $2b+x$, which multiplied by x, the second term of the root, gives $2bx+x^2$, a product that exactly equals the dividend and completes the operation.

---

[*] The author takes it for granted that the student who has progressed to this point is already familiar with the arithmetical processes of extracting the square and cube roots of numbers. Hence, he has thought it unnecessary to introduce the subject into the present work. Those who desire information on these points will find the subject fully treated in the author's *Common School* and *Higher Analytical Arithmetics.*

Ex. 2. Extract the square root of $4x^4+12x^3+x^2-12x+4$.

Analysis. The square root of $4x^4$ is $2x^2$, which we place in the quotient as the first term of the root. We then subtract its square (viz. $4x^4$) from the polynomial, and bring down two terms (viz. $12x^3+x^2$) as a new dividend.

OPERATION.

$$\begin{array}{r|l}
 & 4x^4+12x^3+x^2-12x+4\,(2x^2+3x-2 \\
 & \underline{4x^4} \\
4x^2+3x) & 12x^3+x^2 \\
 & \underline{12x^3+9x^2} \\
4x^2+6x-2) & -8x^2-12x+4 \\
 & \underline{-8x^2-12x+4}
\end{array}$$

Doubling $2x^2$, the first term of the root, we obtain $4x^2$ as the *trial divisor;* dividing $12x^3$ (the first term of the new dividend) by $4x^2$, we get $3x$ as the *second* term of the root, which we place in the quotient, and also annex to the trial divisor, making $4x^2+3x$ as the *complete* divisor. We then multiply the complete divisor, $4x^2+3x$, by $3x$, and subtracting the product from the last dividend, and bringing down the remaining terms, we have $-8x^2-12x+4$ as the last dividend. We now double the terms of the root already found (viz. $2x^2+3x$), and obtain $4x^2+6x$ for a new *trial* divisor.

Dividing $-8x^2$ (the first term of the last dividend) by $4x^2$ (the first term of the *trial* divisor), we obtain -2 as the *third* term of the root, which we place in the quotient and also annex to the *trial* divisor, making the complete divisor, $4x^2+6x-2$. Multiplying the complete divisor by -2, and subtracting the product from the last dividend, we see that nothing is left.

Hence, to extract the square root we have the following

RULE.—*I. Arrange the terms according to the powers of one of the letters; take the square root of the first term, and write its root in the quotient for the first term of the required root; subtract its square from the given polynomial, and bring down two other terms for a dividend.*

II. Double the root already found for a trial divisor; see how often the trial divisor is contained in the first term of the dividend, and place the result in the quotient as the second term of the root, and also annex it to the trial divisor to form the complete divisor.

III. Multiply the complete divisor by the second term of the root, subtract the product from the dividend, and to the remainder bring down as many terms as may be necessary for a new dividend.

IV. Double the terms of the root already found for a NEW TRIAL DIVISOR, *and continue the operation as before until all the periods have been brought down.*

Proof. Square the root, and if the result is equal to the given quantity, the root is correct.

EXAMPLES.

3. What is the square root of $4x^2+12xy+9y^2$? *Ans.* $2x+3y$.

4. Required the square root of $16a^6-40a^3b^2+25b^4$. *Ans.* $4a^3-5b^2$.

5. Find the square root of $b^2+2bc+c^2+2bx+2cx+x^2$. *Ans.* $b+c+x$.

6. What is the square root of $x^4+4x^3-6x^2-20x+25$? *Ans.* x^2+2x-5.

7. Required the square root of $9x^2-12ax+4a^2+6x-4a+1$. *Ans.* $3x-2a+1$.

8. Find the square root of $4x^4-12x^3+25x^2-24x+16$. *Ans.* $2x^2-3x+4$.

9. What is the square root of $9x^4-12x^3+40x^2-24x+36$? *Ans.* $3x^2-2x+6$.

10. Find the square root of $16a^4-24a^3+25a^2-12a+4$. *Ans.* $4a^2-3a+2$.

11. Find the square root of $9b^6-12b^5+34b^4-20b^3+25b^2$. *Ans.* $3b^3-2b^2+5b$.

12. What is the square root of $x^6+4x^5+10x^4+14x^3+13x^2+6x+1$? *Ans.* x^3+2x^2+3x+1.

13. Required the square root of $16x^4+24x^3+81x^2+54x+81$. *Ans.* $4x^2+3x+9$.

CASE III.

193. To Extract the Cube Root of a Polynomial.

To extract the cube root of a quantity is to resolve it into three equal factors; or, it is to find a quantity which multiplied into itself twice will produce the given quantity.

194. We shall better understand the process of extracting the cube root if we analyze what takes place in forming a cube by the multiplication of its factors.

In the series of powers of a binomial, as given in **175**, we find the cube of $(a+b)$ to be as follows:

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

From which we derive this general truth—viz.:

The cube of a binomial is equal to the cube of the first term, +3 times the square of the first multiplied by the second, +3 times the first multiplied by the square of the second, + the cube of the second.

195. Now, suppose we are required to find the cube root of $a^3+3a^2b+3ab^2+b^3$.

OPERATION.

$$\begin{array}{lll|l} & & a^3+3a^2b+3ab^2+b^3 & \underline{a+b} \\ & & \underline{a^3} & \\ \text{TRIAL DIVISOR,} & 3a^2 & & 3a^2b+3ab^2+b^3 \\ \text{COMPLETE DIVISOR,} & 3a^2+3ab+b^2 & & \underline{3a^2b+3ab^2+b^3} \end{array}$$

Analysis. In the formation of the cube we have just seen that a^3 is the *cube* of the *first* term of the root; therefore, the *cube root* of a^3, which is a, will give the *first* term of the root. We place this root on the right. Subtracting the cube of (a) from the polynomial, we obtain $3a^2b+3ab^2+b^3$ as the remainder or dividend. This remainder is obviously equal to 3 *times the square of the first term multiplied by the second,* + 3 *times the first multiplied by the square of the second,* + *the cube of the second.* We already have the *first* term of the root, and our object now is to find the *second.* By way of *trial,* we divide $3a^2b$, the first term of the dividend, by $3a^2$; *i. e.* by 3 *times the square of the first term of the root.* The quotient is b, which we write as the *second* term of the root. The dividend, $3a^2b+3ab^2+b^3$, in addition to 3 times the square of the first term multiplied by the second, also contains 3 times the product of the first term by the square of the second, + the cube of the second. We therefore complete the divisor by adding to the *trial divisor*, $3a^2$, 3 times the product of the first term by the second (*i. e.* $3ab$); and to this we add the square of the second term; *i. e.* b^2. Then, $3a^2+3ab+b^2=$ the *complete divisor.*

Multiplying the complete divisor by b, the second term of the root, and subtracting the product, we find no remainder, which proves that $(a+b)$ is the correct root.

Ex. 1. Find the cube root of $b^3+3b^2x+3bx^2+x^3$.

OPERATION.

$$\begin{array}{lr|l}
 & b^3+3b^2x+3bx^2+x^3 & \underline{b+x} \\
 & b^3 & \\
\hline
\text{T. D. } 3b^2 & \multicolumn{1}{|l}{3b^2x+3bx^2+x^3} & \\
\text{C. D. } 3b^2+3bx+x^2 & \multicolumn{1}{|l}{\underline{3b^2x+3bx^2+x^3}} &
\end{array}$$

Analysis. The cube root of the first term, b^3, is b; which we write in the quotient as the first term of the root. We then subtract the cube of b from the given polynomial, and obtain $3b^2x+3bx^2+x^3=(3b^2+3bx+x^2)x$ as the remainder or dividend. We square the first term of the root and multiply it by 3, making $3b^2$, and write it at the left of the dividend for a TRIAL DIVISOR. We now divide $3b^2x$, the first term of the dividend, by the trial divisor, $3b^2$, and obtain x as the second term of the root. To form the *complete* divisor, we add to $3b^2$ (the trial divisor) 3 times the product of the first term of the root by the second, and also the square of the second. We shall then have $3b^2+3bx+x^2$ as the complete divisor. Multiplying the complete divisor by the last term of the root, we obtain $3b^2x+3bx^2+x^3$; a product that exactly equals the dividend and completes the work.

NOTE.—The explanation above given will apply *generally*, without regard to the number of terms in the root.

2. What is the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$?

OPERATION.

$$\begin{array}{l|l}
 & x^6-6x^5+15x^4-20x^3+15x^2-6x+1 \;\; \underline{x^2-2x+1} \\
 & x^6 \\
\hline
\text{1st. T. D.} & \\
3x^4 & -6x^5+15x^4-20x^3 \\
\text{1st. C. D.} & \\
3x^4-6x^3+4x^2 & \underline{-6x^5+12x^4-\;\;8x^3} \\
\text{2d T. D.} & \\
3(x^2-2x)^2=3x^4-12x^3+12x^2 & 3x^4-12x^3+15x^2-6x+1 \\
\qquad\qquad\qquad\qquad\quad 3x^2-6x+1 & \\
\text{2d C. D.}\quad \overline{3x^4-12x^3+15x^2-6x+1} & \underline{3x^4-12x^3+15x^2-6x+1}
\end{array}$$

Analysis. The cube root of x^6 is x^2, which we write in the quotient as the first term of the root. We subtract the cube of x^2 from the polynomial, and bring down *three* terms, $-6x^5+15x^4-20x^3$, as a dividend. We square the first term of the root, and multiply it by 3, making $3x^4$, and write it for a *trial* divisor. We now divide $-6x^5$, the first term of the dividend, by the trial divisor, $3x^4$, and obtain $-2x$ as the second term of the root. To form the *complete* divisor, we add to $3x^4$ (the trial divisor) 3 times the product of the first term of the root by the second, and also the square of the second. We then have $3x^4-6x^3+4x^2$ as the complete divisor. Multiplying the complete divisor by $-2x$, the second term of the root, we obtain $-6x^5+12x^4-8x^3$. Subtracting this product and bringing down the remaining terms, we have $3x^4-12x^3+15x^2-6x+1$ as a new dividend.

To get the next TRIAL DIVISOR, we take 3 times the square of the two terms already found—viz. $3(x^2-2x)^2=3x^4-12x^3+12x^2$. We now divide $3x^4$, the first term of the new dividend, by $3x^4$, the first term of the new divisor, and obtain $+1$ as the third term of the root. To form the *complete* divisor, we add to $3x^4-12x^3+12x^2$ (the trial divisor) 3 times the product of the first two terms by the third (viz. $3x^2-6x$), and also the square of the last term—viz. $+1$. We then have for the complete divisor $3x^4-12x^3+15x^2-6x+1$. Multiplying the complete divisor by the last term of the root, we have $3x^4-12x^3+15x^2-6x+1$, and subtracting this from the dividend, we find there is no remainder. The root therefore is x^2-2x+1.

From the foregoing explanations we derive the following

RULE.—*I. Arrange the polynomial with reference to the powers of some letter.*

II. Take the cube root of the first term, and place it on the right as the first term of the root; subtract its cube from the polynomial, and bring down the next three terms for a dividend.

III. Square the first term of the root, multiply the result by 3, and place it on the left as a TRIAL DIVISOR; *divide the first term of the dividend by the trial divisor, and write the quotient as the second term of the root. To obtain the* COMPLETE DIVISOR, *add to the trial divisor 3 times the prod-*

uct of the first term of the root by the second, and also the square of the second.

IV. Multiply the complete divisor by the last term of the root, subtract the product from the dividend, and bring down as many terms as may be necessary to continue the operation.

V. Square the whole root already found, multiply the result by 3 for a new trial divisor, and continue the operation as before until all the terms of the polynomial have been brought down.

EXAMPLES.

3. What is the cube root of $a^3-3a^2x+3ax^2-x^3$?
Ans. $a-x$.

4. Find the cube root of $x^6+3x^4y^2+3x^2y^4+y^6$. *Ans.* x^2+y^2.

5. What is the cube root of $8x^3+36x^2y+54xy^2+27y^3$?
Ans. $2x+3y$.

6. Required the cube root of $27x^3+135x^2+225x+125$.
Ans. $3x+5$.

7. What is the cube root of $x^6-6x^5+18x^4-32x^3+36x^2-24x+8$? *Ans.* x^2-2x+2.

8. Find the cube root of $x^6+6x^5+21x^4+44x^3+63x^2+54x+27$.
Ans. x^2+2x+3.

SECTION XIV.

RADICAL QUANTITIES.

196. A RADICAL QUANTITY is a quantity under a radical sign, or a quantity with a fractional index; thus, $\sqrt{2}$, $\sqrt[3]{x}$, $(3)^{\frac{1}{2}}$, $(y)^{\frac{1}{3}}$, etc. are radical quantities.

197. Radicals are divided into different classes, distinguished according to their *degrees*. The degree of a radical quantity is denoted by the index of the radical sign; thus,

$\sqrt{2}$, $\sqrt{a}$ and $(5)^{\frac{1}{2}}$ are radicals of the *second* degree;
$\sqrt[3]{x}$, $\sqrt[3]{4}$ and $(6)^{\frac{1}{3}}$ " " *third* "
$\sqrt[4]{a}$, $\sqrt[4]{7}$ and $(x)^{\frac{1}{4}}$ " " *fourth* "
$\sqrt[n]{x}$, $\sqrt[n]{2}$ and $(y)^{\frac{1}{n}}$ " " *n*th "

198. SIMILAR radicals are those which have the *same* quantity under the *same* radical sign or under the *same* fractional index. Thus, $2\sqrt{3}$ and $5\sqrt{3}$ are *similar* radicals; so are $4\sqrt[3]{a}$ and $7\sqrt[3]{a}$; so are $2a^{\frac{1}{2}}$ and $5a^{\frac{1}{2}}$.

199. A letter or numeral prefixed to the radical sign is called a *coefficient;* thus, $3\sqrt{x}$ shows that the square root of x is taken 3 times, and it is read: "Three times the square root of x;" $5\sqrt[3]{2}$ denotes that the cube root of 2 is taken 5 times, and is read: "Five times the cube root of 2."

When no coefficient is prefixed 1 is always understood.

200. The method of indicating roots by the radical sign is altogether arbitrary; any other symbol would have answered equally as well; but their indication by means of fractional indices is much more convenient and philosophical. It brings *roots* under the same laws with *powers*, and enables us to refer the operations upon radicals to the general principles which govern the operations upon rational quantities.

201. In **191** it was shown that in order to extract any root of a monomial we must divide the exponent of each literal factor by the index of the required root. Thus, the square root of x^4 is x^2; the square root of x^2 is x^1 or x; the square root of x^1 is $x^{\frac{1}{2}}$; hence,

$$x^{\frac{1}{2}} \text{ is equivalent to } \sqrt{x}.$$

Again: the cube root of x^9 is x^3; the cube root of x^3 is x^1; and the cube root of x^1 is $x^{\frac{1}{3}}$; hence,

$$x^{\frac{1}{3}} \text{ is equivalent to } \sqrt[3]{x}.$$

Again: The fourth root of x^{16} is x^4; the fourth root of x^4 is x^1; and the fourth root of x^1 is $x^{\frac{1}{4}}$; hence,

$$x^{\frac{1}{4}} \text{ is equivalent to } \sqrt[4]{x}.$$

In the same manner it can be shown that

$$x^{\frac{1}{5}} \text{ is equivalent to } \sqrt[5]{x},$$
$$x^{\frac{1}{6}} \text{ is equivalent to } \sqrt[6]{x},$$
$$x^{\frac{1}{n}} \text{ is equivalent to } \sqrt[n]{x}.$$

A root of any quantity is therefore properly expressed by writing the quantity under a fractional exponent whose numerator is 1, and whose denominator indicates the number of equal factors into which the quantity is resolved.

202. The cube root of x^6 is x^2, and the cube root of x^2 is $x^{\frac{2}{3}}$. This fractional exponent indicates both a *power* and a root; *i. e.* it denotes the cube *root* of the second *power* of x, or the second power of the cube root of x. The denominator shows that the quantity denoted by x is resolved into three equal factors, and the numerator indicates that *two* of these equal factors are multiplied together, making the *second power* of the cube root of x. One of these equal factors would be expressed by $x^{\frac{1}{3}}$ and the product of the two would be $x^{\frac{1}{3}}x^{\frac{1}{3}} = x^{\frac{1}{3}\times 2} = x^{\frac{2}{3}}$.

Thus, suppose $x = 8$; then, $x^{\frac{2}{3}} = (8)^{\frac{2}{3}}$; *i. e.* the second power of the cube root of 8. But the cube root of 8 is 2, and the second power of 2 is 4; therefore, $(8)^{\frac{2}{3}} = 4$. We might have reached the same result by considering $(8)^{\frac{2}{3}}$ as the cube root of the second power of 8. The second power of 8 is 64, and the cube root of 64 is 4. The result is the same as before.

203. One very important fact connected with fractional indices is, that any index may be exchanged for any other index of the same value; thus,

$$a^{\frac{1}{2}} = a^{\frac{2}{4}} = a^{\frac{3}{6}} = a^{\frac{4}{8}} = a^{\frac{5}{10}}, \text{ etc.};$$

i. e. the square root of any quantity is equal to the fourth root of the second power of that quantity, or to the sixth root of the third power, or to the eighth root of the fourth power, and so on.

For a like reason, $a^{\frac{1}{3}}=a^{\frac{2}{6}}=a^{\frac{3}{9}}=a^{\frac{4}{12}}$, etc.

204. It is obvious that $x^1=x^{\frac{2}{2}}=x^{\frac{3}{3}}=x^{\frac{4}{4}}=x^{\frac{n}{n}}$, etc.; *i. e.* any quantity is equal to the second power of the second root of that quantity, or to the third power of the third root, or to the fourth power of the fourth root, or to the *n*th power of the *n*th root, and so on.

As a natural consequence, whenever a quantity is under the sign of the square root, and we square it or raise it to the second power, this process removes the radical sign, and thus renders the quantity rational; if it is under the sign of the cube root, and we cube it or raise it to the third power, this operation removes the radical sign, and thus renders the quantity rational; thus, if $\sqrt{x}=5$, and we square both sides, we have $x=25$; if $x^{\frac{1}{3}}=2$, then, cubing both sides, we obtain $x=8$; or if $\sqrt[n]{x}=b$, then, raising both sides to the *n*th power, we find $x=b^n$.

As n may represent *any root* whatever, we see that the principle holds good universally; hence,

If any quantity under a simple radical sign be raised to a power of the same name as the corresponding root, this process will remove the radical sign, and thus render the quantity rational.

205. There are various modes of writing these powers of roots, as well as of reading them.

Thus, $(x)^{\frac{4}{5}}$ may be written as follows:

$$(x^4)^{\frac{1}{5}},\text{ or }(x^{\frac{1}{5}})^4,\text{ or }\sqrt[5]{x^4},\text{ or }(\sqrt[5]{x})^4.$$

NOTE.—*To Teachers.* The future progress of the student will be expedited by his being required to write and to read examples like the foregoing in all the various forms under which they may be represented.

Ex. 1. Express the fourth root of the third power of x.

Ans. $(x^3)^{\frac{1}{4}}$, or $(x^{\frac{1}{4}})^3$, or $\sqrt[4]{x^3}$, or $(\sqrt[4]{x})^3$, or $x^{\frac{3}{4}}$.

How is the first of these expressions read?

Ans. The *fourth* root of the *third* power of x.

How is the second expression read?

Ans. The *third* power of the *fourth* root of x.

What does the denominator 4 of the fractional index show?

Ans. It shows that the quantity represented by x has been separated into *four equal factors.*

How would *one* of those equal factors be indicated?

Ans. By $x^{\frac{1}{4}}$ or $\sqrt[4]{x}$.

What does the numerator 3 of the index denote?

Ans. It shows that *three* of the equal factors into which x has been separated are to be multiplied together; thus,

$$x^{\frac{1}{4}} \times x^{\frac{1}{4}} \times x^{\frac{1}{4}} = x^{\frac{1}{4} \times 3} = x^{\frac{3}{4}}.$$

Questions like the above will readily suggest themselves to the teacher, and their effect will be to dissipate much of the fogginess which, in the mind of a beginner, is apt to hang around the subject of radicals.

EXAMPLES.

2. Express the fifth root of the third power of x.

Ans. $(x^3)^{\frac{1}{5}}$, or $(x^{\frac{1}{5}})^3$, or $\sqrt[5]{x^3}$, or $(\sqrt[5]{x})^3$, or $x^{\frac{3}{5}}$.

3. Express the third power of the fourth root of $(a+b)$.

Ans. ——.

4. Express the fifth root of the fourth power of 10.

Ans. ——.

5. Express the fourth power of the fifth root of $(a-b)$

Ans. ——.

6. Express the sixth root of the fifth power of 12.

Ans. ——.

7. Express the fifth power of the sixth root of $(a+m)$.

Ans. ——.

8. Express the seventh root of the third power of 4.

Ans. ——.

9. Show that $(x)^{\frac{2}{3}}$ is equal to 9 when $x=27$.

Analysis. Here we are required to find the second power of the cube root of 27. The cube root of 27 is 3, and the second power of 3 is $3\times3=9$ *Ans.*

NOTE.—It will generally be more convenient to find the *root* first, and then raise the root to the *power* required.

10. Show that $(a)^{\frac{3}{4}}$ is equal to 8 when $a=16$.

11. Show that $(x)^{\frac{5}{2}}$ is equal to 32 when $x=4$.

12. Show that $(b)^{\frac{3}{2}}$ is equal to 125 when $b=25$.

13. Show that $(c)^{\frac{4}{3}}$ is equal to 256 when $c=64$.

14. Show that $(n)^{\frac{5}{3}}$ is equal to 32 when $n=8$.

15. Show that $(x)^{\frac{4}{3}}$ is equal to 81 when $x=27$.

16. Show that $(y)^{\frac{3}{2}}$ is equal to 27 when $y=9$.

17. Show that $(m)^{\frac{3}{4}}$ is equal to 27 when $m=81$.

18. Show that $(z)^{\frac{5}{2}}$ is equal to 243 when $z=9$.

19. Show that $(x)^{\frac{3}{2}}$ is equal to 343 when $x=49$.

REDUCTION OF RADICALS.

206. The REDUCTION OF RADICALS consists in changing their *form* without changing their *value.*

CASE I.

207. To Remove a Part of a Root from under the Radical Sign.

Ex. 1. Remove a factor from $\sqrt{18}$.

Analysis. The greatest square contained in 18 is 9. We therefore resolve 18 into the two factors 9 and 2. But $\sqrt{9}=3$. We then substitute 3 instead of $\sqrt{9}$, and get $3\sqrt{2}$.

OPERATION.

$$\begin{aligned}\sqrt{18}&=\sqrt{9\times 2}\\&=\sqrt{9}\times\sqrt{2}\\&=3\sqrt{2}\end{aligned}$$

NOTE.—This process is generally called reducing a radical quantity to *its simplest form.*

2. Reduce $\sqrt[3]{81}$ to its simplest form.

Analysis. The greatest cube in 81 is 27. We resolve 81 into the two factors 27 and 3. But $\sqrt[3]{27}=3$. We then substitute 3 in place of $\sqrt[3]{27}$, and we have $3\sqrt[3]{3}$.

OPERATION.

$$\begin{aligned}\sqrt[3]{81}&=\sqrt[3]{27\times 3}\\&=\sqrt[3]{27}\times\sqrt[3]{3}\\&=3\sqrt[3]{3}\end{aligned}$$

In both of the foregoing examples we have changed the *form* of the radical, but have not altered the *value.*

3. Reduce $\sqrt{75a^3b^5}$ to its simplest form.

Analysis. The greatest square in $75a^3b^5$ is $25a^2b^4$. We then resolve $75a^3b^5$ into the two factors, $25a^2b^4$ and $3ab$. But $\sqrt{25a^2b^4}=5ab^2$. We then substitute $5ab^2$ in place of $\sqrt{25a^2b^4}$, and we have $5ab^2\sqrt{3ab}$.

OPERATION.

$$\begin{aligned}\sqrt{75a^3b^5}&=\sqrt{25a^2b^4\times 3ab}\\&=\sqrt{25a^2b^4}\times\sqrt{3ab}\\&=5ab^2\sqrt{3ab}\end{aligned}$$

From the foregoing illustrations we derive the following

RULE—*Separate the given quantity into two factors, one of which is the greatest* EXACT *power of the same name with the root, and the other the* IMPERFECT *power. Find the root of this exact power, and prefix it to the other factor, with the radical sign between them.*

EXAMPLES.

4. Reduce $\sqrt{50}$ to its simplest form. *Ans.* $5\sqrt{2}$.
5. Reduce $\sqrt[3]{54}$ to its simplest form. *Ans.* $3\sqrt[3]{2}$.
6. Reduce $\sqrt{72}$ to its simplest form. *Ans.* $6\sqrt{2}$.
7. Reduce $\sqrt[3]{192}$ to its simplest form. *Ans.* $4\sqrt[3]{3}$.
8. Reduce $\sqrt{48x^3}$ to its simplest form. *Ans.* $4x\sqrt{3x}$.
9. Reduce $(40a^3b^4)^{\frac{1}{3}}$ to its simplest form. *Ans.* $2ab(5b)^{\frac{1}{3}}$.

10. Reduce $(72a^3y^5)^{\frac{1}{2}}$ to its simplest form.

Ans. $6ay^2(2ay)^{\frac{1}{2}}$.

11. Reduce $(128a^4b^5)^{\frac{1}{3}}$ to its simplest form.

Ans. $4ab(2ab^2)^{\frac{1}{3}}$.

12. Reduce $\sqrt{98a^2x}$ to its simplest form. Ans. $7a\sqrt{2x}$.

13. Reduce $\sqrt[3]{135a^3y^2}$ to its simplest form. Ans. $3a\sqrt[3]{5y^2}$.

14. Reduce $3\sqrt{32a^3}$ to its simplest form.

OPERATION.

$$3\sqrt{32a^3}=3\sqrt{16a^2\times 2a}=3\sqrt{16a^2}\times\sqrt{2a}=3\times 4a\sqrt{2a}=12a\sqrt{2a}.$$

Ans. $12a\sqrt{2a}$.

15. Reduce $5\sqrt{20a^3}$ to its simplest form. Ans. $10a\sqrt{5a}$.

16. Reduce $3\sqrt[3]{192y^4}$ to its simplest form. Ans. $12y\sqrt[3]{3y}$.

17. Reduce $4(108ay^2)^{\frac{1}{2}}$ to its simplest form.

Ans. $24y(3ay)^{\frac{1}{2}}$.

18. Reduce $5(81a^3y^4)^{\frac{1}{3}}$ to its simplest form.

Ans. $15ay(3y)^{\frac{1}{3}}$

19. Reduce $\sqrt{x^3-ax^2+x^2}$ to its simplest form.

Analysis. By inspection we see that the greatest square in $(x^3-ax^2+x^2)$ is x^2. We therefore resolve $x^3-ax^2+x^2$ into the two factors, x^2 and $(x-a+1)$. But $\sqrt{x^2}=x$. We then substitute x for $\sqrt{x^2}$, and have $x\sqrt{x-a+1}$.

OPERATION.

$$\begin{aligned}\sqrt{x^3-ax^2+x^2}&=\sqrt{x^2(x-a+1)}\\&=\sqrt{x^2}\times\sqrt{x-a+1}\\&=x\sqrt{x-a+1}\end{aligned}$$

20. Reduce $(3y^3-y^4)^{\frac{1}{3}}$ to its simplest form.

Analysis. We see that the greatest *cube* in $(3y^3-y^4)$ is y^3. Therefore, we resolve $(3y^3-y^4)$ into the two factors y^3 and $(3-y)$. But $(y^3)^{\frac{1}{3}}=y$, and substituting y for $(y^3)^{\frac{1}{3}}$, we have $y(3-y)^{\frac{1}{3}}$.

OPERATION.

$$\begin{aligned}(3y^3-y^4)^{\frac{1}{3}}&=[y^3\times(3-y)]^{\frac{1}{3}}\\&=(y^3)^{\frac{1}{3}}\times(3-y)^{\frac{1}{3}}\\&=y(3-y)^{\frac{1}{3}}\end{aligned}$$

21. Reduce $\sqrt{y^3-y^2b}$ to its simplest form. Ans. $y\sqrt{y-b}$.

22. Reduce $\sqrt[3]{a^4-a^3b}$ to its simplest form. *Ans.* $a\sqrt[3]{a-b}$.

23. Reduce $\sqrt{a^2x-a^2+a^3}$ to its simplest form.
Ans. $a\sqrt{x-1+a}$.

24. Reduce $\sqrt[3]{a^4+a^3-a^3y}$ to its simplest form.
Ans. $a\sqrt[3]{a+1-y}$.

25. Reduce $(8y^2-12y^3)^{\frac{1}{2}}$ to its simplest form.
Ans. $2y(2-3y)^{\frac{1}{2}}$.

26. Reduce $(16a^3-8a^4)^{\frac{1}{3}}$ to its simplest form.
Ans. $2a(2-a)^{\frac{1}{3}}$.

27. Reduce $\sqrt{\dfrac{18a^3}{48y^5}}$ to its simplest form.

OPERATION.

$$\sqrt{\frac{18a^3}{48y^5}}=\sqrt{\frac{9a^2\times 2a}{16y^4\times 3y}}$$
$$=\sqrt{\frac{9a^2}{16y^4}}\times\sqrt{\frac{2a}{3y}}$$
$$=\frac{3a}{4y^2}\sqrt{\frac{2a}{3y}}$$

Analysis. We resolve both numerator and denominator into their proper factors, just as in the case of integral quantities, and place the reduced numerator over the reduced denominator.

28. Reduce $\sqrt{\frac{32}{75}}$ to its simplest form. *Ans.* $\frac{4}{5}\sqrt{\frac{2}{3}}$.

29. Reduce $\sqrt{\dfrac{y^4b}{a^2x}}$ to its simplest form. *Ans.* $\dfrac{y^2}{a}\sqrt{\dfrac{b}{x}}$.

30. Reduce $\sqrt[3]{\dfrac{a^3y}{b^4c}}$ to its simplest form. *Ans.* $\dfrac{a}{b}\sqrt[3]{\dfrac{y}{bc}}$.

31. Reduce $\sqrt{\dfrac{4b^3}{25y^5}}$ to its simplest form. *Ans.* $\dfrac{2b}{5y^2}\sqrt{\dfrac{b}{y}}$.

32. Reduce $\sqrt[3]{\dfrac{8a^4}{27x^5}}$ to its simplest form. *Ans.* $\dfrac{2a}{3x}\sqrt[3]{\dfrac{a}{x^2}}$.

208. When the radical is in the form of a *fraction* it may often be simplified by multiplying both numerator and denominator by some quantity that will make the denominator a *perfect* power of the degree required.

Ex. 1. Reduce $\sqrt{\frac{5}{8}}$ to its simplest form.

OPERATION.

$$\sqrt{\tfrac{5}{8}}=\sqrt{\tfrac{10}{16}}$$
$$=\sqrt{\tfrac{1}{16}\times 10}$$
$$=\sqrt{\tfrac{1}{16}}\times\sqrt{10}$$
$$=\tfrac{1}{4}\sqrt{10}$$

Analysis. We multiply both numerator and denominator by 2, which does not alter the *value* of the fraction (**125**), but makes the denominator a *perfect square.* We then resolve the fraction into two factors, one of which is the greatest square; then, extracting the square root of this square, we prefix it to the other factor, with the radical sign between them.

NOTE.—The student will readily understand that $\frac{1}{4}\sqrt{10}$ is a more *simple* expression than $\sqrt{\frac{5}{8}}$, because, in order to obtain the numerical value of $\sqrt{\frac{5}{8}}$, it would be necessary to extract the square root of 5, and also of 8, and then divide one of these roots by the other; whereas in the expression $\frac{1}{4}\sqrt{10}$ it is only necessary to extract the square root of 10, and multiply this root by $\frac{1}{4}$.

2. Find the simplest form of $\sqrt[3]{\frac{7}{9}}$.

OPERATION.

$$\sqrt[3]{\tfrac{7}{9}}=\sqrt[3]{\tfrac{21}{27}}$$
$$=\sqrt[3]{\tfrac{1}{27}\times 21}$$
$$=\sqrt[3]{\tfrac{1}{27}}\times\sqrt[3]{21}$$
$$=\tfrac{1}{3}\sqrt[3]{21}$$

Analysis. We multiply both numerator and denominator by 3, because this will make the denominator a *perfect cube.* We then resolve the fraction into two factors, one of which is the greatest cube, and, extracting the cube root of this cube, we prefix it to the other factor, just as in the case of integral quantities.

3. Find the simplest form of $\frac{2}{5}\sqrt[3]{\frac{3}{4}}$.

OPERATION.

$$\tfrac{2}{5}\sqrt[3]{\tfrac{3}{4}}=\tfrac{2}{5}\sqrt[3]{\tfrac{6}{8}}=\tfrac{2}{5}\sqrt[3]{\tfrac{1}{8}\times 6}=\tfrac{2}{5}\times\tfrac{1}{2}\sqrt[3]{6}=\tfrac{2}{10}\sqrt[3]{6}=\tfrac{1}{5}\sqrt[3]{6}.$$

NOTE.—It is obvious that we may make the denominator of the fraction a perfect power by multiplying it by itself the requisite number of times, but we can frequently shorten the process by multiplying by some smaller number. Thus, in the last example we might have made the denominator, 4, a perfect *cube* by multiplying it into itself *twice*, making it 64; but we shortened the labor by simply multiplying by 2, which gave us 8, a perfect cube. A little practice will soon render this familiar.

4. Reduce $\sqrt{\frac{1}{3}}$ to its simplest form. *Ans.* $\frac{1}{3}\sqrt{3}$.

5. Reduce $\sqrt[3]{\frac{1}{4}}$ to its simplest form. *Ans.* $\frac{1}{2}\sqrt[3]{2}$.

6. Find the simplest form of $\sqrt{\frac{1}{2}}$. *Ans.* $\frac{1}{2}\sqrt{2}$.

7. Find the simplest form of $\sqrt[3]{\frac{1}{2}}$. *Ans.* $\frac{1}{2}\sqrt[3]{4}$.

8. Find the simplest form of $\sqrt{\frac{2}{3}}$. *Ans.* $\frac{1}{3}\sqrt{6}$.

9. Find the simplest form of $\sqrt[3]{\frac{5}{9}}$. *Ans.* $\frac{1}{3}\sqrt[3]{15}$.

10. Reduce $\sqrt{\frac{5}{12}}$ to its simplest form. *Ans.* $\frac{1}{6}\sqrt{15}$.

11. Reduce $\sqrt[3]{\frac{3}{16}}$ to its simplest form. *Ans.* $\frac{1}{4}\sqrt[3]{12}$.

12. Reduce $3\sqrt{\frac{1}{5}}$ to its simplest form. *Ans.* $\frac{3}{5}\sqrt{5}$.

13. Reduce $3\sqrt[3]{\frac{5}{32}}$ to its simplest form. *Ans.* $\frac{3}{4}\sqrt[3]{10}$.

14. Reduce $\frac{2}{3}\sqrt{\frac{7}{8}}$ to its simplest form. *Ans.* $\frac{1}{6}\sqrt{14}$.

15. Reduce $\frac{5}{7}\sqrt[3]{\frac{2}{3}}$ to its simplest form. *Ans.* $\frac{5}{21}\sqrt[3]{18}$.

16. Reduce $\frac{5}{6}\sqrt{\frac{3}{5}}$ to its simplest form. *Ans.* $\frac{1}{6}\sqrt{15}$.

17. Reduce $\frac{2}{3}\sqrt[3]{\frac{1}{2}}$ to its simplest form. *Ans.* $\frac{1}{3}\sqrt[3]{4}$.

18. Reduce $\frac{3}{4}\sqrt{\frac{49}{147}}$ to its simplest form. *Ans.* $\frac{1}{4}\sqrt{3}$.

CASE II.

209. To Reduce a RATIONAL Quantity to the Form of a Radical.

Ex. 1. Reduce $5a$ to the form of the square root.

Analysis. Any quantity is equal to the square root of its square; hence, $5a$ is equal to the square root of $(5a)^2$, which equals the square root of $25a^2 = \sqrt{25a^2}$. The quantity is reduced to the *form* of a radical without any alteration of *value*.

OPERATION.

$$\begin{aligned} 5a &= (5a)^{\frac{2}{2}} \text{ (204)} \\ &= [(5a)^2]^{\frac{1}{2}} \\ &= (25a^2)^{\frac{1}{2}} \\ &= \sqrt{25a^2} \end{aligned}$$

2. Reduce $3x$ to the form of the cube root.

Analysis. Any quantity is equal to the cube root of its cube; hence, $3x$ is equal to the cube root of $(3x)^3$; which equals the cube root of $27x^3$, which equals $\sqrt[3]{27x^3}$; $3x$ is reduced to the *form* of a radical, while its *value* is unchanged.

OPERATION.

$$\begin{aligned} 3x &= (3x)^{\frac{3}{3}} \\ &= [(3x)^3]^{\frac{1}{3}} \\ &= (27x^3)^{\frac{1}{3}} \\ &= \sqrt[3]{27x^3} \end{aligned}$$

From the foregoing illustrations we derive the following

RULE.—*Raise the quantity to a* POWER *of the same name with the given root, and place over it the corresponding radical sign or fractional index.*

EXAMPLES.

3. Reduce $6y$ to the form of the square root.

Ans. $\sqrt{36y^2}$.

4. Reduce $5b$ to the form of the cube root.

Ans. $\sqrt[3]{125b^3}$.

5. Reduce $7a^2$ to the form of the square root.

Ans. $\sqrt{49a^4}$.

6. Reduce $3x^2$ to the form of the cube root.

Ans. $\sqrt[3]{27x^6}$.

7. Reduce $6a^2x^3$ to the form of the square root.

Ans. $\sqrt{36a^4x^6}$.

8. Reduce $4a^2y^3$ to the form of the cube root.

Ans. $\sqrt[3]{64a^6y^9}$.

9. Reduce $3\sqrt{2y}$ to the form of the square root.

OPERATION.

$$3\sqrt{2y}=\sqrt{3^2}\times\sqrt{2y}=\sqrt{9}\times\sqrt{2y}=\sqrt{9\times 2y}=\sqrt{18y}.$$

10. Reduce $2\sqrt{3a}$ to the form of the square root.

Ans. $\sqrt{12a}$.

11. Reduce $3\sqrt[3]{2x}$ to the form of the cube root.

Ans. $\sqrt[3]{54x}$.

12. Reduce $5a\sqrt{3y}$ to the form of the square root.

Ans. $\sqrt{75a^2y}$.

13. Reduce $4x\sqrt[3]{5a}$ to the form of the cube root.

Ans. $\sqrt[3]{320ax^3}$.

14. Reduce $6a\sqrt{2a^3}$ to the form of the square root.

Ans. $\sqrt{72a^5}$.

15. Reduce $5y\sqrt[3]{3y}$ to the form of the cube root.

Ans. $\sqrt[3]{375y^4}$.

CASE III.

210. To Reduce Radicals which have Different Indices to Equivalent Radicals having a Common Index.

Ex. 1. Reduce $a^{\frac{1}{3}}$ and $b^{\frac{1}{4}}$ to a common index.

Analysis. From (**208**) it is obvious that the index of a power or root may be exchanged for any other index having the same value; hence, $a^{\frac{1}{3}}$ may be exchanged for $a^{\frac{4}{12}}$, and $b^{\frac{1}{4}}$ for $b^{\frac{3}{12}}$.

OPERATION.

$$a^{\frac{1}{3}} = a^{\frac{4}{12}} = (a^4)^{\frac{1}{12}} = \sqrt[12]{a^4}$$
$$b^{\frac{1}{4}} = b^{\frac{3}{12}} = (b^3)^{\frac{1}{12}} = \sqrt[12]{b^3}$$

2. Reduce $(3)^{\frac{1}{2}}$ and $(4)^{\frac{1}{3}}$ to a common index.

OPERATION.

$$(3)^{\frac{1}{2}} = (3)^{\frac{3}{6}} = [(3)^3]^{\frac{1}{6}} = (27)^{\frac{1}{6}} = \sqrt[6]{27}$$
$$(4)^{\frac{1}{3}} = (4)^{\frac{2}{6}} = (4^2)^{\frac{1}{6}} = (16)^{\frac{1}{6}} = \sqrt[6]{16}$$

3. Reduce $(4)^{\frac{1}{3}}$ and $(5)^{\frac{1}{4}}$ to a common index.

OPERATION.

$$(4)^{\frac{1}{3}} = (4)^{\frac{4}{12}} = (4^4)^{\frac{1}{12}} = (256)^{\frac{1}{12}} = \sqrt[12]{256}$$
$$(5)^{\frac{1}{4}} = (5)^{\frac{3}{12}} = (5^3)^{\frac{1}{12}} = (125)^{\frac{1}{12}} = \sqrt[12]{125}$$

From the preceding illustrations we derive the following

RULE.—*I. Reduce the indices to a common denominator.*

II. Raise each quantity to the power expressed by the numerator of its reduced index.

III. Express the root denoted by the denominator.

EXAMPLES.

4. Reduce $a^{\frac{1}{2}}$ and $y^{\frac{1}{5}}$ to a common index.

Ans. $(a^5)^{\frac{1}{10}}$ and $(y^2)^{\frac{1}{10}}$.

5. Reduce $x^{\frac{1}{3}}$ and $y^{\frac{1}{4}}$ to a common index.

Ans. $(x^4)^{\frac{1}{12}}$ and $(y^3)^{\frac{1}{12}}$.

6. Reduce $(2)^{\frac{1}{2}}$ and $(3)^{\frac{1}{3}}$ to a common index.

Ans. $(8)^{\frac{1}{6}}$ and $(9)^{\frac{1}{6}}$.

7. Reduce $(ax)^{\frac{1}{3}}$ and $(by)^{\frac{1}{4}}$ to a common index.

Ans. $(a^4x^4)^{\frac{1}{12}}$ and $(b^3y^3)^{\frac{1}{12}}$.

8. Reduce $(5)^{\frac{1}{2}}$ and $(6)^{\frac{1}{3}}$ to a common index.

Ans. $(125)^{\frac{1}{6}}$ and $(36)^{\frac{1}{6}}$.

9. Reduce $(cx)^{\frac{2}{3}}$ and $(my)^{\frac{1}{2}}$ to a common index.

Ans. $(c^4x^4)^{\frac{1}{6}}$ and $(m^3y^3)^{\frac{1}{6}}$.

10. Reduce $(2)^{\frac{1}{3}}$ and $(3)^{\frac{1}{4}}$ to a common index.

Ans. $(16)^{\frac{1}{12}}$ and $(27)^{\frac{1}{12}}$.

ADDITION OF RADICALS.

211. Ex. 1. What is the sum of $4\sqrt{x}$ and $6\sqrt{x}$?

OPERATION.

$$4\sqrt{x}+6\sqrt{x}=10\sqrt{x}$$

Analysis. Here the radical parts are *alike;* and it is just as evident that 4 times the square root of x and 6 times the square root of x make 10 times the square root of x, as that 4 apples and 6 apples make 10 apples.

2. What is the sum of $3\sqrt{12}$ and $5\sqrt{27}$?

OPERATION.

$$3\sqrt{12}=3\sqrt{4\times3}=3\times2\sqrt{3}=\ 6\sqrt{3}$$
$$5\sqrt{27}=5\sqrt{9\times3}=5\times3\sqrt{3}=15\sqrt{3}$$
$$\text{Sum}=21\sqrt{3}$$

Analysis. Here the radical parts are *unlike*, but by reducing them to their simple form we make them *alike*, and then add their coefficients. Thus, $3\sqrt{12}$ was found to be equal to $6\sqrt{3}$, and $5\sqrt{27}=15\sqrt{3}$, and their sum is equal to $21\sqrt{3}$.

3. Find the sum of $4\sqrt[3]{54}$ and $5\sqrt[3]{128}$.

OPERATION.

$$4\sqrt[3]{54}=4\sqrt[3]{27\times2}=4\times3\sqrt[3]{2}=12\sqrt[3]{2}$$
$$5\sqrt[3]{128}=5\sqrt[3]{64\times2}=5\times4\sqrt[3]{2}=20\sqrt[3]{2}$$
$$\text{Sum}=32\sqrt[3]{2}$$

Analysis. Here the radical parts are *unlike*, but by reduction they are made *alike*, and are then added, just as rational quantities are added.

From the foregoing explanations we derive the following

RULE.—*I. Reduce the radicals to their most simple form.*

II. If the radical parts are alike, add their coefficients, and to their sum annex the common radical part.

EXAMPLES.

4. What is the sum of $3\sqrt{32}$ and $2\sqrt{18}$? *Ans.* $18\sqrt{2}$.
5. Find the sum of $3\sqrt{18}$ and $4\sqrt{8}$. *Ans.* $17\sqrt{2}$.
6. What is the sum of $3\sqrt[3]{81}$ and $2\sqrt[3]{24}$? *Ans.* $13\sqrt[3]{3}$.
7. What is the sum of $\sqrt{108}$ and $\sqrt{147}$? *Ans.* $13\sqrt{3}$.
8. Find the sum of $\sqrt[3]{108}$ and $\sqrt[3]{256}$. *Ans.* $7\sqrt[3]{4}$.
9. What is the sum of $\sqrt{192}$ and $\sqrt{243}$? *Ans.* $17\sqrt{3}$.
10. Find the sum of $5\sqrt[3]{16}$ and $4\sqrt[3]{54}$. *Ans.* $22\sqrt[3]{2}$.
11. What is the sum of $3\sqrt{45}$ and $4\sqrt{80}$? *Ans.* $25\sqrt{5}$.
12. Find the sum of $\sqrt[3]{500}$ and $\sqrt[3]{256}$. *Ans.* $9\sqrt[3]{4}$.
13. What is the sum of $\sqrt{98a^2x}$ and $\sqrt{72a^2x}$? *Ans.* $13a\sqrt{2x}$.
14. Find the sum of $\sqrt[3]{54b^4y}$ and $\sqrt[3]{16b^4y}$. *Ans.* $5b\sqrt[3]{2by}$.
15. What is the sum of $\sqrt{18x}$ and $\sqrt{2b^2x}$? *Ans.* $(3+b)\sqrt{2x}$.
16. Find the sum of $\sqrt[3]{24ay}$ and $\sqrt[3]{3a^4y}$. *Ans.* $(2+a)\sqrt[3]{3ay}$.
17. What is the sum of $\sqrt{a^3x}$ and $\sqrt{ab^2x}$? *Ans.* $(a+b)\sqrt{ax}$.
18. What is the sum of $\sqrt{18}$ and $\sqrt{12}$?

Analysis. Here, after reduction, the radical parts are still *unlike*, and their addition can only be indicated by *writing them one after another, with their proper signs.*

OPERATION.

$\sqrt{18}=\sqrt{9\times2}=3\sqrt{2}$

$\sqrt{12}=\sqrt{4\times3}=2\sqrt{3}$

$\text{Sum}=\overline{3\sqrt{2}+2\sqrt{3}}$

19. Find the sum of $\sqrt{20}$ and $\sqrt{27}$. *Ans.* $2\sqrt{5}+3\sqrt{3}$.

20. What is the sum of $\sqrt[3]{24}$ and $\sqrt[3]{16}$?

Ans. $2\sqrt[3]{3}+2\sqrt[3]{2}$.

21. Find the sum of $3\sqrt{27}$ and $2\sqrt{\frac{1}{3}}$.

Analysis. By reduction the radical parts are made *alike*, and are then added as in previous examples.

OPERATION.

$$\begin{aligned}3\sqrt{27} &= 3\sqrt{9\times 3} = 3\times 3\sqrt{3} &&= 9\sqrt{3}\\ 2\sqrt{\tfrac{1}{3}} &= 2\sqrt{\tfrac{3}{9}} = 2\sqrt{\tfrac{1}{9}\times 3} = 2\times\tfrac{1}{3}\sqrt{3} &&= \tfrac{2}{3}\sqrt{3}\\ & &&\text{Sum} = 9\tfrac{2}{3}\sqrt{3}\end{aligned}$$

22. What is the sum of $\sqrt{18}$ and $6\sqrt{\frac{1}{2}}$? *Ans.* $6\sqrt{2}$.

23. Find the sum of $\sqrt[3]{16}$ and $4\sqrt[3]{\frac{1}{4}}$. *Ans.* $4\sqrt[3]{2}$.

SUBTRACTION OF RADICALS.

212. Ex. 1. Find the difference between $5\sqrt{18}$ and $3\sqrt{8}$.

Analysis. We first reduce the radicals to their simplest form, thus making them *alike*—viz. $5\sqrt{18}=15\sqrt{2}$ and $3\sqrt{8}=6\sqrt{2}$; then $6\sqrt{2}$ from $15\sqrt{2}$ leaves $9\sqrt{2}$.

OPERATION.

$$\begin{aligned}5\sqrt{18} &= 5\sqrt{9\times 2} = 5\times 3\sqrt{2} = 15\sqrt{2}\\ 3\sqrt{8} &= 3\sqrt{4\times 2} = 3\times 2\sqrt{2} = 6\sqrt{2}\\ &\text{Difference} = 9\sqrt{2}\end{aligned}$$

2. Find the difference between $3\sqrt[3]{81}$ and $4\sqrt[3]{192}$.

Analysis. We reduce $4\sqrt[3]{192}$ to $16\sqrt[3]{3}$, and $3\sqrt[3]{81}$ to $9\sqrt[3]{3}$, thus making the radical parts *alike*. Then $9\sqrt[3]{3}$ from $16\sqrt[3]{3}$ evidently leaves $7\sqrt[3]{3}$.

OPERATION.

$$\begin{aligned}4\sqrt[3]{192} &= 4\sqrt[3]{64\times 3} = 4\times 4\sqrt[3]{3} = 16\sqrt[3]{3}\\ 3\sqrt[3]{81} &= 3\sqrt[3]{27\times 3} = 3\times 3\sqrt[3]{3} = 9\sqrt[3]{3}\\ &\text{Difference} = 7\sqrt[3]{3}\end{aligned}$$

From the foregoing explanations we derive the following

RULE.—*I. Reduce the radicals to their simplest form.*

II. If the radicals are alike, subtract their coefficients, and to their difference annex the common radical part.

III. If the radical parts after reduction are not alike, indicate their difference by the proper sign.

EXAMPLES.

3. Find the difference between $9\sqrt{7}$ and $5\sqrt{7}$. *Ans.* $4\sqrt{7}$.

4. What is the difference between $7y\sqrt{ax}$ and $3y\sqrt{ax}$? *Ans.* $4y\sqrt{ax}$.

5. From $a\sqrt{x}$ subtract $b\sqrt{x}$. *Ans.* $(a-b)\sqrt{x}$.

6. From $n\sqrt[3]{y}$ subtract $5\sqrt[3]{y}$. *Ans.* $(n-5)\sqrt[3]{y}$.

7. Subtract $\sqrt{12}$ from $\sqrt{27}$. *Ans.* $\sqrt{3}$.

8. Take $\sqrt{75}$ from $\sqrt{147}$. *Ans.* $2\sqrt{3}$.

9. Subtract $\sqrt[3]{54}$ from $\sqrt[3]{128}$. *Ans.* $\sqrt[3]{2}$.

10. From $3\sqrt{50}$ take $2\sqrt{32}$. *Ans.* $7\sqrt{2}$.

11. From $5\sqrt[3]{192}$ take $3\sqrt[3]{81}$. *Ans.* $11\sqrt[3]{3}$.

12. From $\sqrt{192}$ take $\sqrt{75}$. *Ans.* $3\sqrt{3}$.

13. Take $\sqrt[3]{135}$ from $\sqrt[3]{320}$. *Ans.* $\sqrt[3]{5}$.

14. From $5\sqrt{48}$ take $3\sqrt{27}$. *Ans.* $11\sqrt{3}$.

15. From $5\sqrt[3]{54}$ take $5\sqrt[3]{16}$. *Ans.* $5\sqrt[3]{2}$.

16. Take $2\sqrt{50}$ from $4\sqrt{72}$. *Ans.* $14\sqrt{2}$.

17. From $5\sqrt{y}$ take $3\sqrt{x}$. *Ans.* $5\sqrt{y}-3\sqrt{x}$.

MULTIPLICATION OF RADICALS.

213. One important principle lies at the foundation of all operations in the multiplication of radical quantities—viz.:

The root of a product of several factors is equal to the product of their roots.

Suppose we wish to multiply $\sqrt{9}$ by $\sqrt{4}$. The operation would be thus; $\sqrt{9}\times\sqrt{4}=\sqrt{36}=6=$ *root of their product.* But $\sqrt{9}=3$, and $\sqrt{4}=2$, and $3\times2=6=$ *product of their roots.*

Hence we see that *the root of the product gives the same result as the product of the roots.*

Ex. 1. Multiply $3\sqrt{2}$ by $5\sqrt{3}$.

Analysis. $\sqrt{2}$ multiplied by $\sqrt{3}=\sqrt{6}$; then, $3\sqrt{2}$ multiplied by $\sqrt{3}$ gives a product *three* times as great $=3\sqrt{6}$; and $3\sqrt{2}$ multiplied by $5\sqrt{3}$ gives a product *five* times as great $=15\sqrt{6}$.

OPERATION.
Multiplicand, $3\sqrt{2}$
Multiplier, $5\sqrt{3}$
Product $=15\sqrt{6}$

2. Multiply $4\sqrt{10}$ by $3\sqrt{5}$.

Analysis. $\sqrt{10}$ multiplied by $\sqrt{5}=\sqrt{50}$; then, $4\sqrt{10}$ multiplied by $\sqrt{5}$ gives a product 4 *times* as great $=4\sqrt{50}$; and $4\sqrt{10}$ multiplied by $3\sqrt{5}$ gives *three* times as much $=12\sqrt{50}$; which reduced gives $60\sqrt{2}$.

OPERATION.
Multiplicand, $4\sqrt{10}$
Multiplier, $3\sqrt{5}$
Product $=12\sqrt{50}=12\sqrt{25\times 2}$
$=12\times 5\sqrt{2}$
$=60\sqrt{2}$

From the foregoing Analyses we derive the following

RULE.—*When the quantities are under the same radical sign or index, multiply the rational coefficients together, and prefix their product to the product of the radical parts.*

EXAMPLES.

3. Multiply $4\sqrt{8}$ by $5\sqrt{3}$. *Ans.* $20\sqrt{24}=40\sqrt{6}$

4. What is the product of $5\sqrt{2}$ and $3\sqrt{6}$? *Ans.* $15\sqrt{12}=30\sqrt{3}$.

5. Multiply $7\sqrt{5}$ by $4\sqrt{2}$. *Ans.* $28\sqrt{10}$.

6. What is the product of $4\sqrt{8}$ and $3\sqrt{2}$? *Ans.* 48.

7. Multiply $6\sqrt{3}$ by $5\sqrt{6}$. *Ans.* $90\sqrt{2}$.

8. What is the product of $4\sqrt{7}$ and $5\sqrt{7}$? *Ans.* 140.

9. Multiply $5\sqrt{8}$ by $3\sqrt{6}$. *Ans.* $60\sqrt{3}$.

10. Multiply $3\sqrt[3]{9}$ by $4\sqrt[3]{6}$. *Ans.* $12\sqrt[3]{54}=36\sqrt[3]{2}$.

11. Multiply $3\sqrt[3]{3}$ by $5\sqrt[3]{9}$. *Ans.* 45.

12. Multiply $6\sqrt[3]{4}$ by $2\sqrt[3]{6}$. *Ans.* $24\sqrt[3]{3}$.

13. Multiply $5\sqrt[3]{7}$ by $4\sqrt[3]{3}$. *Ans.* $20\sqrt[3]{21}$.

14. Multiply $5\sqrt{ab}$ by $3\sqrt{ax}$. *Ans.* $15a\sqrt{bx}$.

15. Multiply $3\sqrt[3]{a^2y}$ by $4\sqrt[3]{ay}$. *Ans.* $12a\sqrt[3]{y^2}$.

16. Multiply $x\sqrt{a^2y}$ by $b\sqrt{ay^3}$. *Ans.* $abxy\sqrt{ay}$.

17. Multiply $2\sqrt[3]{a+b}$ by $3\sqrt[3]{a-b}$. *Ans.* $6\sqrt[3]{a^2-b^2}$.

18. Multiply $a(c+x)^{\frac{1}{2}}$ by $b(c-x)^{\frac{1}{2}}$. *Ans.* $ab(c^2-x^2)^{\frac{1}{2}}$.

19. Multiply $c(a+x)^{\frac{1}{3}}$ by $y(a-x)^{\frac{1}{3}}$. *Ans.* $cy(a^2-x^2)^{\frac{1}{3}}$.

20. Multiply $a\sqrt{y}$ by $b\sqrt{y}$. *Ans.* aby.

21. Multiply $2\sqrt{5}$ by $6\sqrt{5}$. *Ans.* 60.

22. Multiply $(3)^{\frac{1}{2}}$ by $(4)^{\frac{1}{3}}$.

Analysis. As the radical indices are not the same, we reduce the radical quantities to the same index (**210**), thus changing $(3)^{\frac{1}{2}}$ to $(27)^{\frac{1}{6}}$ and $(4)^{\frac{1}{3}}$ to $(16)^{\frac{1}{6}}$; and now the quantities, being under the *same* index, can be multiplied, just as in the preceding examples.

OPERATION.

$$\begin{array}{lr} \text{Multiplicand,} & (3)^{\frac{1}{2}}=(3)^{\frac{3}{6}}=(3^3)^{\frac{1}{6}}= \quad (27)^{\frac{1}{6}} \\ \text{Multiplier,} & (4)^{\frac{1}{3}}=(4)^{\frac{2}{6}}=(4^2)^{\frac{1}{6}}= \underline{\quad (16)^{\frac{1}{6}}} \\ & \text{Product}=(432)^{\frac{1}{6}} \end{array}$$

23. Multiply $\sqrt[3]{a}$ by $\sqrt[4]{b}$.

Analysis. We exchange the radical signs for fractional indices. We then reduce the radical quantities to the same index; thus changing $(a)^{\frac{1}{3}}$ to $(a^4)^{\frac{1}{12}}$ and $(b)^{\frac{1}{4}}$ to $(b^3)^{\frac{1}{12}}$; and now the quantities, being under the *same* index, can be multiplied just as rational quantities, the product being placed under the common index, $\frac{1}{12}$.

OPERATION.

$$\begin{array}{lr} \text{Multiplicand,} & \sqrt[3]{a}=(a)^{\frac{1}{3}}=(a)^{\frac{4}{12}}= \quad (a^4)^{\frac{1}{12}} \\ \text{Multiplier,} & \sqrt[4]{b}=(b)^{\frac{1}{4}}=(b)^{\frac{3}{12}}= \underline{\quad (b^3)^{\frac{1}{12}}} \\ & \text{Product}=(a^4b^3)^{\frac{1}{12}} \end{array}$$

24. Multiply $\sqrt{2}$ by $\sqrt[3]{3}$. *Ans.* $\sqrt[6]{72}=(72)^{\frac{1}{6}}$.

25. Multiply $\sqrt{x}$ by $\sqrt[3]{y}$. *Ans.* $\sqrt[6]{x^3y^2}=(x^3y^2)^{\frac{1}{6}}$.

26. Multiply $a\sqrt[3]{b}$ by $c\sqrt[4]{d}$. *Ans.* $ac\sqrt[12]{b^4d^3}=ac(b^4d^3)^{\frac{1}{12}}$.

27. Multiply $(ax)^{\frac{1}{2}}$ by $(cy)^{\frac{1}{3}}$. *Ans.* $\sqrt[6]{a^3x^3c^2y^2}=(a^3x^3c^2y^2)^{\frac{1}{6}}$.

28. Multiply $\sqrt{a}$ by $\sqrt[3]{a}$.

Analysis. Here the *quantities* under the radical sign being *alike*, but the radical indices being unlike, we reduce the indices to a common denominator, and add them, or multiply just as in previous examples.

OPERATION.

$$\begin{array}{l} \sqrt{a} = (a)^{\frac{1}{2}} = a^{\frac{3}{6}} = (a^3)^{\frac{1}{6}} \\ \sqrt[3]{a} = (a)^{\frac{1}{3}} = \underline{a^{\frac{2}{6}} = (a^2)^{\frac{1}{6}}} \\ \text{Product} = a^{\frac{5}{6}} = (a^5)^{\frac{1}{6}} \end{array}$$

29. Multiply $\sqrt[3]{y}$ by $\sqrt[4]{y}$. *Ans.* $(y)^{\frac{7}{12}}$, or $(y^7)^{\frac{1}{12}}$.

30. Multiply $\sqrt{ab}$ by $\sqrt[3]{ab}$. *Ans.* $(ab)^{\frac{5}{6}} = (a^5b^5)^{\frac{1}{6}}$.

31 Multiply $3\sqrt{2}$ by $5\sqrt[3]{2}$.
Ans. $15(2)^{\frac{5}{6}} = 15(2^5)^{\frac{1}{6}} = 15(32)^{\frac{1}{6}}$.

32. Multiply $(a+x)^{\frac{1}{2}}$ by $(a+x)^{\frac{1}{3}}$. *Ans.* $(a+x)^{\frac{5}{6}} = \overline{(a+x)^5}|^{\frac{1}{6}}$.

33. Multiply $(b-y)^{\frac{1}{3}}$ by $(b-y)^{\frac{1}{4}}$.
Ans. $(b-y)^{\frac{7}{12}} = \overline{(b-y)^7}|^{\frac{1}{12}}$.

34. Multiply $a(x-y)^{\frac{1}{2}}$ by $c(x-y)^{\frac{1}{3}}$.
Ans. $ac(x-y)^{\frac{5}{6}} = ac\overline{(x-y)^5}|^{\frac{1}{6}}$.

35. Multiply $a+\sqrt{x}$ and $b+\sqrt{x}$.

Analysis. When the radical quantities are connected with others by the signs + or −, the multiplication is performed by multiplying each term of the multiplicand by each term of the multiplier (**59**), just as other compound quantities are multiplied.

OPERATION.

$$\begin{array}{ll} \text{Multiplicand} & a+\sqrt{x} \\ \text{Multiplier,} & \underline{b+\sqrt{x}} \\ & ab+b\sqrt{x} \\ & \underline{\quad +a\sqrt{x}+\sqrt{x^2}} \\ \text{Product} = & ab+a\sqrt{x}+b\sqrt{x}+x \end{array}$$

36. Multiply $3+\sqrt{5}$ and $6-\sqrt{5}$.

Analysis. Here we multiply each term of the multiplicand by each term of the multiplier, placing like terms in the same column, and then uniting the results.

OPERATION.

$$\begin{array}{ll} \text{Multiplicand,} & 3+\sqrt{5} \\ \text{Multiplier,} & \underline{6-\sqrt{5}} \\ & 18+6\sqrt{5} \\ & \underline{\quad -3\sqrt{5}-\sqrt{25}} \\ \text{Product} & =18+3\sqrt{5}-\sqrt{25} \\ & =18+3\sqrt{5}-5 \\ & =13+3\sqrt{5} \end{array}$$

37. Multiply $(a+\sqrt{b})$ by $(c-\sqrt{d})$.
Ans. $ac+c\sqrt{b}-a\sqrt{d}-\sqrt{bd}$.

38. Multiply $(x+\sqrt{y})$ by $(a+\sqrt{y})$.

Ans. $ax+a\sqrt{y}+x\sqrt{y}+y$.

39. Multiply $(3+\sqrt{6})$ by $(5+\sqrt{6})$.

Ans. $15+8\sqrt{6}+6=21+8\sqrt{6}$.

40. Multiply $(4+\sqrt{3})$ and $(4-\sqrt{3})$. *Ans.* 13.

41. Multiply $(5+\sqrt{x})$ and $(5+\sqrt{x})$.

Ans. $25+10\sqrt{x}+x$.

42. Multiply $(3+2\sqrt{x})$ and $(3+2\sqrt{x})$.

Ans. $9+12\sqrt{x}+4x$.

43. Multiply $(\sqrt{a}+\sqrt{x})$ and $(\sqrt{a}+\sqrt{x})$.

Ans. $a+2\sqrt{ax}+x$.

44. Multiply $(7-2\sqrt{5})$ and $(7+4\sqrt{5})$. *Ans.* $9+14\sqrt{5}$.

DIVISION OF RADICALS.

214. The division of radical quantities depends upon the principle that

The root of the quotient of two quantities is the same as the quotient of their roots.

Thus, if we were required to divide $\sqrt{36}$ by $\sqrt{4}$, the operation would be as follows:

$$\sqrt{36}\div\sqrt{4}=\frac{\sqrt{36}}{\sqrt{4}}=\sqrt{\tfrac{36}{4}}=\sqrt{9}=3=\textit{the root of the quotient.}$$

But $\sqrt{36}=6$ and $\sqrt{4}=2$; hence, $6\div2=\frac{6}{2}=3$ = the *quotient* of the roots.

Ex. 1. Divide $\sqrt{72}$ by $\sqrt{6}$.

OPERATION.

$$\sqrt{72}\div\sqrt{6}=\frac{\sqrt{72}}{\sqrt{6}}=\sqrt{\tfrac{72}{6}}=\sqrt{12}=\sqrt{4\times3}=2\sqrt{3}.\quad \textit{Ans. } 2\sqrt{3}$$

2. Divide $12\sqrt{126}$ by $4\sqrt{7}$.

OPERATION.

$$12\sqrt{126}\div4\sqrt{7}=\frac{12\sqrt{126}}{4\sqrt{7}}=3\sqrt{\tfrac{126}{7}}=3\sqrt{18}=3\sqrt{9\times2}=3\times3\sqrt{2}=9\sqrt{2}$$

Ans. $9\sqrt{2}$.

From the foregoing illustrations we derive the following

RULE.—*When the quantities are under the same radical sign or index, divide the coefficient of the dividend by the coefficient of the divisor, and prefix the quotient to the quotient of the radical parts.*

EXAMPLES.

3. Divide $\sqrt{108}$ by $\sqrt{12}$. *Ans.* $\sqrt{9}=3$.
4. Divide $\sqrt[3]{256}$ by $\sqrt[3]{4}$. *Ans.* $\sqrt[3]{64}=4$.
5. Divide $8\sqrt{35}$ by $2\sqrt{7}$. *Ans.* $4\sqrt{5}$.
6. Divide $\sqrt{54a^2x}$ by $\sqrt{18ax}$. *Ans.* $\sqrt{3a}$.
7. Divide $15\sqrt{54}$ by $3\sqrt{6}$. *Ans.* 15.
8. Divide $18\sqrt[3]{45}$ by $3\sqrt[3]{9}$. *Ans.* $6\sqrt[3]{5}$.
9. Divide $14\sqrt[3]{48}$ by $7\sqrt[3]{6}$. *Ans.* $2\sqrt[3]{8}=4$.
10 Divide $16\sqrt{108}$ by $4\sqrt{3}$. *Ans.* 24.
11. Divide $a^2x\sqrt{by}$ by $a\sqrt{y}$. *Ans.* $ax\sqrt{b}$.
12. Divide $12\sqrt{a^2-x^2}$ by $3\sqrt{a-x}$. *Ans.* $4\sqrt{a+x}$.
13. Divide $20\sqrt[3]{162}$ by $5\sqrt[3]{3}$. *Ans.* $12\sqrt[3]{2}$.
14. Divide $27\sqrt{192}$ by $9\sqrt{3}$. *Ans.* 24.
15. Divide $15\sqrt[3]{108}$ by $5\sqrt[3]{4}$. *Ans.* 9.
16. Divide $9ax\sqrt{a^4}$ by $3x\sqrt{a^2}$. *Ans.* $3a^2$.
17. Divide $10ay\sqrt[3]{by^2}$ by $5a\sqrt[3]{y}$. *Ans.* $2y\sqrt[3]{by}$.
18. Divide $12a(8y)^{\frac{1}{2}}$ by $3(2y)^{\frac{1}{2}}$. *Ans.* $8a$.
19. Divide $(x^2-y^2)^{\frac{1}{3}}$ by $(x+y)^{\frac{1}{3}}$. *Ans.* $(x-y)^{\frac{1}{3}}$.
20. Divide $8(a^2+2ab+b^2)^{\frac{1}{2}}$ by $2(a+b)^{\frac{1}{2}}$. *Ans.* $4(a+b)^{\frac{1}{2}}$.
21. Divide $\sqrt{ax}$ by $\sqrt[3]{a}$.

Analysis. We first exchange the radical signs for fractional indices. We then reduce the indices to a common index, then divide as in preceding examples, placing the quotient under the common index.

OPERATION.

$$\sqrt{ax}=(ax)^{\frac{1}{2}}=(ax)^{\frac{3}{6}}=(a^3x^3)^{\frac{1}{6}},$$

$$\sqrt[3]{a}=(a)^{\frac{1}{3}}=(a)^{\frac{2}{6}}=(a^2)^{\frac{1}{6}};$$

$$\text{then } \frac{(a^3x^3)^{\frac{1}{6}}}{(a^2)^{\frac{1}{6}}}=\left(\frac{a^3x^3}{a^2}\right)^{\frac{1}{6}}=(ax^3)^{\frac{1}{6}}.$$

22. Divide $\sqrt{by}$ by $\sqrt[3]{y}$. *Ans.* $(b^3y)^{\frac{1}{6}}$, or $\sqrt[6]{b^3y}$.

23. Divide $\sqrt[3]{ax}$ by $\sqrt[4]{a}$. *Ans.* $(ax^4)^{\frac{1}{12}}$, or $\sqrt[12]{ax^4}$.

24. Divide $15\sqrt{mx}$ by $3\sqrt[3]{m}$. *Ans.* $5(mx^3)^{\frac{1}{6}}$, or $5\sqrt[6]{mx^3}$.

25. Divide $20\sqrt[3]{ac}$ by $4\sqrt[5]{a}$. *Ans.* $5(a^2c^5)^{\frac{1}{15}}$, or $5\sqrt[15]{a^2c^5}$.

26. Divide $\sqrt{x}$ by $\sqrt[3]{x}$.

Analysis. Here the quantities are the same, but the radicals are of different degrees. We first exchange the radical signs for fractional indices, then reduce the indices to a *common* index, and subtract the index of the divisor from the index of the dividend, just as in the division of *powers* (**75**).

OPERATION.

$$\begin{aligned} \text{Dividend} &= \sqrt{x} = (x)^{\frac{1}{2}} = (x)^{\frac{3}{6}} \\ \text{Divisor} &= \sqrt[3]{x} = (x)^{\frac{1}{3}} = \underline{(x)^{\frac{2}{6}}} \\ \text{Quotient} &= (x)^{\frac{1}{6}} \end{aligned}$$

27. Divide $\sqrt{2}$ by $\sqrt[3]{2}$. *Ans.* $(2)^{\frac{1}{6}} = \sqrt[6]{2}$.

28. Divide $\sqrt[3]{mx}$ by $\sqrt[4]{mx}$. *Ans.* $(mx)^{\frac{1}{12}} = \sqrt[12]{mx}$.

29. Divide $8(ay)^{\frac{1}{2}}$ by $4(ay)^{\frac{1}{3}}$. *Ans.* $2(ay)^{\frac{1}{6}}$.

30. Divide $9(5)^{\frac{1}{2}}$ by $3(5)^{\frac{1}{4}}$. *Ans.* $3(5)^{\frac{1}{4}}$.

31. Divide $10(3)^{\frac{1}{3}}$ by $2(3)^{\frac{1}{4}}$. *Ans.* $5(3)^{\frac{1}{12}}$.

32. Divide $12(axy)^{\frac{1}{4}}$ by $6(axy)^{\frac{1}{5}}$. *Ans.* $2(axy)^{\frac{1}{20}}$.

INVOLUTION OF RADICALS.

215. The general principles which apply to the involution of rational quantities apply also to the involution of radical quantities.

Ex. 1. Involve $3\sqrt{x}$ to the third power.

OPERATION.

$$(3\sqrt{x})^3 = 3\sqrt{x} \times 3\sqrt{x} \times 3\sqrt{x} = 27\sqrt{x^3} = 27\sqrt{x^2 \times x} = 27x\sqrt{x}$$

Or we might express it thus:

$$[3(x)^{\frac{1}{2}}]^3 = 3^3 \times (x^{\frac{1}{2}})^3 = 27(x)^{\frac{3}{2}} = 27(x^3)^{\frac{1}{2}} = 27(x^2 \times x)^{\frac{1}{2}} = 27x(x)^{\frac{1}{2}}$$

2. Raise $4(2)^{\frac{1}{2}}$ to the third power.

OPERATION.

$$[4(2)^{\frac{1}{2}}]^3 = 4^3(2)^{\frac{3}{2}} = 64(2^3)^{\frac{1}{2}} = 64(8)^{\frac{1}{2}} = 64(4 \times 2)^{\frac{1}{2}} = 64 \times 2(2)^{\frac{1}{2}} = 128(2)^{\frac{1}{2}}$$

From the foregoing illustrations we deduce the following

RULE.—*Raise the coefficient to the required power, and prefix it to the required power of the radical part.*

EXAMPLES.

3. Raise $3\sqrt{a}$ to the second power. *Ans.* $9a$.

4. Involve $7(a+x)^{\frac{1}{2}}$ to the second power. *Ans.* $49(a+x)$.

5. Raise $4\sqrt{2}$ to the second power. *Ans.* 32.

6. Involve $(a+x)^{\frac{1}{3}}$ to the third power. *Ans.* $a+x$.

7. Raise $\sqrt{x+5}$ to the second power. *Ans.* $x+5$.

8. Involve $\sqrt[3]{3+x}$ to the third power. *Ans.* $3+x$.

NOTE.—Any root is raised to a power of the same name simply by removing the radical sign or fractional index.

9. Raise $5+\sqrt{x}$ to the second power.

Analysis. When the rational parts are connected with the radicals by the signs + or −, we raise them to powers by actual multiplication, as in the case of other polynomials.

OPERATION.

$$\begin{array}{l} \quad 5+\sqrt{x} \\ \quad 5+\sqrt{x} \\ \hline 25+\ \ 5\sqrt{x} \\ \quad +\ \ 5\sqrt{x}+\sqrt{x^2}=x \\ \hline 25+10\sqrt{x}+x \end{array}$$

10. Involve $3+\sqrt{a}$ to the second power. *Ans.* $9+6\sqrt{a}+a$.

11. Raise $5+\sqrt{3}$ to the second power. *Ans.* $28+10\sqrt{3}$.

12. Involve $4-\sqrt{2}$ to the second power. *Ans.* $18-8\sqrt{2}$.

13. Raise $6+2\sqrt{3}$ to the second power. *Ans.* $48+24\sqrt{3}$.

14. Involve $7-3\sqrt{2}$ to the second power. *Ans.* $67-42\sqrt{2}$

EVOLUTION OF RADICALS.

216. As *evolution* is the reverse of *involution*, we may learn the process required to extract a root from observing the operation in obtaining the power.

Suppose we are required to raise $3(a)^{\frac{1}{6}}$ to the cube. The operation would be indicated as follows:

$$(3a^{\frac{1}{6}})^3 = 3^3 \times (a^{\frac{1}{6}})^3 = 27a^{\frac{3}{6}} = 27a^{\frac{1}{2}};$$

from which it is evident that the cube root of $27a^{\frac{1}{2}}$ must be $3a^{\frac{1}{6}}$. But 3 is the *cube root* of 27, and $a^{\frac{1}{6}}$ is the cube root of $a^{\frac{1}{2}}$.

Hence, to extract the root of a radical quantity with a rational coefficient we have the following

RULE.—*Extract the required root of the rational coefficients, and divide the index of the radical by the index of the required root.*

EXAMPLES.

1. What is the square root of $25a^{\frac{1}{2}}$? *Ans.* $\pm 5a^{\frac{1}{4}} = \pm 5\sqrt[4]{a}$.
2. What is the cube root of $8a^{\frac{1}{2}}$? *Ans.* $2a^{\frac{1}{6}} = 2\sqrt[6]{a}$.
3. Extract the square root of $49a^{\frac{1}{3}}$. *Ans.* $\pm 7a^{\frac{1}{6}}$.
4. What is the cube root of $27a^{\frac{1}{4}}$? *Ans.* $3a^{\frac{1}{12}}$.
5. Extract the square root of $4a^{\frac{2}{3}}$. *Ans.* $\pm 2a^{\frac{1}{3}}$.
6. What is the cube root of $-64a^{\frac{1}{3}}$? *Ans.* $-4a^{\frac{1}{9}}$.
7. What is the fourth root of $16a^{\frac{1}{2}}$? *Ans.* $\pm 2a^{\frac{1}{8}}$.
8. Extract the cube root of $125x^{\frac{3}{4}}$. *Ans.* $5x^{\frac{1}{4}}$.
9. What is the square root of $289y^{\frac{1}{4}}$? *Ans.* $\pm 17y^{\frac{1}{8}}$.

PROMISCUOUS EXAMPLES.

1. Reduce $\sqrt{147}$ to its simplest form. *Ans.* $7\sqrt{3}$.

2. Reduce $\sqrt[3]{250}$ to its simplest form. *Ans.* $5\sqrt[3]{2}$.

3 Reduce $3\sqrt{108a^3b}$ to its simplest form.
Ans. $18a\sqrt{3ab}$.

4. Reduce $5\sqrt[3]{54a^2b^4}$ to its simplest form.
Ans. $15b\sqrt[3]{2a^2b}$.

5. Reduce $\sqrt{a^3-a^2+a^2x}$ to its simplest form.
Ans. $a\sqrt{a-1+x}$.

6. Reduce $\sqrt[3]{4x^3-x^4}$ to its simplest form. *Ans.* $x\sqrt[3]{4-x}$.

7. Reduce $\sqrt{\frac{48}{125}}$ to its simplest form. *Ans.* $\frac{4}{5}\sqrt{\frac{3}{5}}$.

8. Reduce $\sqrt[3]{\frac{a^4}{x^5}}$ to its simplest form. *Ans.* $\frac{a}{x}\sqrt[3]{\frac{a}{x^2}}$.

9. Reduce $\sqrt{\frac{1}{5}}$ to its simplest form. *Ans.* $\frac{1}{5}\sqrt{5}$.

10. Reduce $\sqrt[3]{\frac{1}{9}}$ to its simplest form. *Ans.* $\frac{1}{3}\sqrt[3]{3}$.

11. Reduce $\sqrt{\frac{7}{12}}$ to its simplest form. *Ans.* $\frac{1}{6}\sqrt{21}$.

12. Reduce $\sqrt[3]{\frac{5}{16}}$ to its simplest form. *Ans.* $\frac{1}{4}\sqrt[3]{20}$.

13. Reduce $5a^3$ to the form of the square root.
Ans. $\sqrt{25a^6}$.

14. Reduce $6a^2x$ to the form of the cube root.
Ans. $\sqrt[3]{216a^6x^3}$.

15 Reduce $5\sqrt{3y}$ to the form of the square root.
Ans. $\sqrt{75y}$.

16. Reduce $4\sqrt[3]{5a^2}$ to the form of the cube root.
Ans $\sqrt[3]{320a^2}$.

17. Reduce $x^{\frac{1}{2}}$ and $y^{\frac{1}{3}}$ to a common index.
Ans $(x^3)^{\frac{1}{6}}$ and $(y^2)^{\frac{1}{6}}$.

18. Reduce $a^{\frac{1}{3}}$ and $x^{\frac{1}{4}}$ to a common index.
Ans. $(a^4)^{\frac{1}{12}}$ and $(x^3)^{\frac{1}{12}}$.

19. Reduce $(3)^{\frac{1}{3}}$ and $(4)^{\frac{1}{2}}$ to a common index.
Ans. $(9)^{\frac{1}{6}}$ and $(64)^{\frac{1}{6}}$.
20. Reduce $(5)^{\frac{1}{4}}$ and $(2)^{\frac{1}{3}}$ to a common index.
Ans. $(125)^{\frac{1}{12}}$ and $(16)^{\frac{1}{12}}$.
21. What is the sum of $\sqrt{75}$ and $\sqrt{147}$? *Ans.* $12\sqrt{3}$.
22. Find the sum of $\sqrt[3]{54}$ and $\sqrt[3]{128}$. *Ans.* $7\sqrt[3]{2}$.
23. What is the difference between $3\sqrt{98}$ and $2\sqrt{50}$?
Ans. $11\sqrt{2}$.
24 Find the difference between $4\sqrt[3]{54}$ and $5\sqrt[3]{16}$.
Ans. $2\sqrt[3]{2}$.
25. Multiply $4\sqrt{8}$ by $5\sqrt{2}$. *Ans.* 80.
26. Multiply $3\sqrt[3]{6}$ by $5\sqrt[3]{9}$. *Ans.* $45\sqrt[3]{2}$.
27. Multiply $\sqrt{a}$ by $\sqrt{a^3x}$. *Ans.* $a^2\sqrt{x}$.
28. Multiply $\sqrt[3]{a^2x}$ by $\sqrt[3]{ax^2}$. *Ans.* ax.
29. Multiply $\sqrt{x}$ by $\sqrt[3]{y}$. *Ans.* $(x^3y^2)^{\frac{1}{6}}$ or $\sqrt[6]{x^3y^2}$.
30. Multiply $\sqrt[3]{c}$ by $\sqrt[4]{x}$. *Ans.* $(c^4x^3)^{\frac{1}{12}}$.
31. Multiply $x+\sqrt{n}$ by $a+\sqrt{n}$.
Ans. $ax+a\sqrt{n}+x\sqrt{n}+n$.
32. Multiply $3+\sqrt{6}$ by $4-\sqrt{6}$. *Ans.* $6+\sqrt{6}$.
33. Multiply $5+\sqrt{2}$ by $7+\sqrt{2}$. *Ans.* $37+12\sqrt{2}$.
34. Divide $\sqrt{245}$ by $\sqrt{5}$. *Ans.* 7.
35. Divide $\sqrt[3]{243}$ by $\sqrt[3]{9}$. *Ans.* 3.
36. Divide $16\sqrt{192}$ by $4\sqrt{6}$. *Ans.* $16\sqrt{2}$.
37. Divide $20\sqrt[3]{405}$ by $4\sqrt[3]{5}$. *Ans.* $15\sqrt[3]{3}$.
38. Raise $6\sqrt{x}$ to the second power. *Ans.* $36x$.
39 Raise $5\sqrt{a+y}$ to the second power. *Ans.* $25(a+y)$.
40. Involve $4\sqrt[3]{a+b}$ to the third power. *Ans.* $64(a+b)$.
41. Raise $4+\sqrt{y}$ to the second power. *Ans.* $16+8\sqrt{y}+y$.
42. Involve $5-\sqrt{3}$ to the second power.
Ans. $28-10\sqrt{3}$

43. What is the square root of $36\sqrt{a}$?

Ans. $\pm 6a^{\frac{1}{4}} = \pm 6\sqrt[4]{a}$.

44. Find the cube root of $-125a^{\frac{1}{2}}$. *Ans.* $-5a^{\frac{1}{6}}$.

SIMPLE EQUATIONS CONTAINING RADICAL QUANTITIES.

217. Ex. 1. Given $\sqrt{x}+5=8$, to find the value of x.

Analysis. We first transpose 5, and the radical then stands by itself. We then square both members of the equation to get rid of the radical sign, and we have $x=9$.

OPERATION.

$$\begin{aligned}\sqrt{x}+5&=8\\ \sqrt{x}&=3\\ x&=9\end{aligned}$$

2. Given $\sqrt{x+2}+6=9$, to find x.

Analysis. We first transpose 6, and the radical then stands by itself. We then square both members, which takes off the radical sign, and by transposition we find $x=7$.

OPERATION.

$$\begin{aligned}\sqrt{x+2}+6&=9\\ \sqrt{x+2}&=3\\ x+2&=9\\ x&=7\end{aligned}$$

3. Given $8+\sqrt[3]{x-4}=10$, to find x.

Analysis. We first transpose 8 to the second member of the equation, thus making the radical to stand alone. We then *cube* both members, which takes off the radical sign, and reducing, we find $x=12$.

OPERATION.

$$\begin{aligned}8+\sqrt[3]{x-4}&=10\\ \sqrt[3]{x-4}&=2\\ x-4&=8\\ x&=12\end{aligned}$$

EXAMPLES.

4. Given $7+\sqrt{x-5}=9$, to find x. *Ans.* $x=9$.

5. Given $\sqrt{x+9}+7=12$, to find x. *Ans.* $x=16$.

6. Given $7+\sqrt{\frac{x}{5}}=9$, to find x. *Ans.* $x=20$.

7. Given $\sqrt[3]{x-5}+9=12$, to find x. *Ans.* $x=32$.

8. Given $8+4\sqrt{x-3}=20$, to find x. *Ans.* $x=12$.

9. Given $5+3(x+6)^{\frac{1}{2}}=17$, to find x. *Ans.* $x=10$.

10. Given $5+\left(\frac{x+4}{3}\right)^{\frac{1}{3}}=7$, to find x. *Ans.* $x=20$.

11. Given $2+\sqrt{x}=\sqrt{x+16}$, to find x.

Analysis. We first square both sides of the equation, which clears the second member entirely of radicals. We then transpose and unite terms, and get $4\sqrt{x}=12$. Then, dividing by 4, the coefficient of the radical, we get $\sqrt{x}=3$; and squaring both sides again, we find $x=9$.

OPERATION.

$$\begin{aligned} 2+\sqrt{x}&=\sqrt{x+16} \\ 4+4\sqrt{x}+x&=x+16 \\ 4\sqrt{x}&=12 \\ \sqrt{x}&=3 \\ x&=9 \end{aligned}$$

12. Given $1+\sqrt{x}=\sqrt{x+5}$, to find x. *Ans.* $x=4$.

13. Given $\sqrt{27+x}=3+\sqrt{x}$, to find x. *Ans.* $x=9$.

14. Given $\sqrt{x+15}=15-\sqrt{x}$, to find x. *Ans.* $x=49$.

15. Given $\sqrt{x-9}=9-\sqrt{x}$, to find x. *Ans.* $x=25$.

16. Given $\sqrt{2x+12}=6-\sqrt{2x}$, to find x. *Ans.* $x=2$.

17. Given $\sqrt{x-28}=14-\sqrt{x}$, to find x. *Ans.* $x=64$.

18. Given $\sqrt{x-32}=\sqrt{x}-2$, to find x. *Ans.* $x=81$.

19. Given $\sqrt{4x+9}=2\sqrt{x}+1$, to find x. *Ans.* $x=4$.

20. Given $\sqrt{5+x}+\sqrt{x}=\frac{15}{\sqrt{5+x}}$, to find x.

Analysis. We first clear of fractions by multiplying the whole equation by $\sqrt{5+x}$, which gives Eq. (2). We then bring the radical to stand alone on one side, and get Eq. (3). We then square both sides, which frees the equation from radicals, and reducing, we find $x=4$.

OPERATION.

$$\begin{aligned} \sqrt{5+x}+\sqrt{x}&=\frac{15}{\sqrt{5+x}} && (1) \\ 5+x+\sqrt{5x+x^2}&=15 && (2) \\ \sqrt{5x+x^2}&=10-x && (3) \\ 5x+x^2&=100-20x+x^2 && (4) \\ 25x&=100 && (5) \\ x&=4 \end{aligned}$$

21. Given $\sqrt{x+7}+\sqrt{x}=\frac{28}{\sqrt{x+7}}$, to find x. *Ans.* $x=9$.

22. Given $\sqrt{x-5}+\sqrt{x}=\frac{10}{\sqrt{x-5}}$, to find x. *Ans.* $x=9$.

23. Given $\sqrt{x-3}+\sqrt{x}=\frac{3}{\sqrt{x-3}}$, to find x. *Ans.* $x=4$.

24 Given $\sqrt{x+9}+\sqrt{x}=\frac{45}{\sqrt{x+9}}$, to find x. *Ans.* $x=16$

25. Given $\sqrt{x-8}+\sqrt{x}=\frac{4}{\sqrt{x-8}}$, to find x. *Ans.* $x=9$.

26. Given $\sqrt{x+3}+\sqrt{x}=\frac{6}{\sqrt{x+3}}$, to find x. *Ans.* $x=1$.

27. Given $\sqrt{x-1}+\sqrt{x+4}=5$, to find x.

OPERATION.

$$
\begin{aligned}
\sqrt{x-1}+\sqrt{x+4}&=5 &(1)\\
\sqrt{x-1}&=5-\sqrt{x+4} &(2)\\
x-1&=25-10\sqrt{x+4}+x+4 &(3)\\
10\sqrt{x+4}&=30 &(4)\\
\sqrt{x+4}&=3 &(5)\\
x+4&=9 &(6)\\
x&=5
\end{aligned}
$$

Analysis. We first transpose one of the radicals to the right-hand side. We do this because it renders the work more simple. We then square both sides, which frees the first member from radicals. Transposing and uniting terms, we bring the radical to stand by itself on one side. We square both sides again, which removes the radical sign, and reducing, we find $x=5$.

28. Given $\sqrt{x+2}-\sqrt{x-3}=1$, to find x. *Ans.* $x=7$.
29. Given $\sqrt{x+1}+\sqrt{x+6}=5$, to find x. *Ans.* $x=3$.
30. Given $\sqrt{x+10}-\sqrt{x-2}=2$, to find x. *Ans.* $x=6$.
31. Given $\sqrt{x+5}+\sqrt{x+12}=7$, to find x. *Ans.* $x=4$.
32. Given $\sqrt{x+8}+\sqrt{x+3}=5$, to find x. *Ans.* $x=1$.

33. Given $\frac{\sqrt{x}+20}{\sqrt{x}+2}=\frac{\sqrt{x}+4}{\sqrt{x}-2}$, to find x.

Analysis. We first clear the equation of fractions by multiplying each numerator by the denominator of the other. We then transpose and unite terms, so as to bring the radical by itself on one side. We then divide each side by the coefficient of the radical; and now, squaring both sides and reducing, we find $x=16$.

OPERATION.

$$\frac{\sqrt{x}+20}{\sqrt{x}+2}=\frac{\sqrt{x}+4}{\sqrt{x}-2}$$
$$x+18\sqrt{x}-40=x+6\sqrt{x}-8$$
$$12\sqrt{x}=48$$
$$\sqrt{x}=4$$
$$x=16$$

34. Given $\frac{\sqrt{x}+9}{\sqrt{x}+5}=\frac{\sqrt{x}+6}{\sqrt{x}+3}$, to find x. *Ans.* $x=9$.

35. Given $\frac{\sqrt{x}-2}{\sqrt{x}+1}=\frac{\sqrt{x}+1}{\sqrt{x}+7}$, to find x. *Ans.* $x=25$.

36. Given $\frac{\sqrt{x}+14}{\sqrt{x}+2}=\frac{\sqrt{x}+4}{\sqrt{x}-2}$, to find x. *Ans.* $x=36$.

SECTION XV.

QUADRATIC EQUATIONS.

218. EQUATIONS are divided into different classes, called *degrees*, according to the highest power of the unknown quantity which they contain.

Those which contain only the *first* power of the unknown quantity are called *simple* equations, or equations of the *first degree*. Those which contain the *square* of the unknown quantity are called *quadratic* equations, or equations of the *second degree*. There are also equations of the *third*, *fourth*, *fifth*, and higher degrees. Thus,

$ax+b=c+d$ is a simple equation;
$3x^2+2x=25$ is a quadratic equation;
$x^3+ax^2+bx=c$ is a cubic equation.

219. QUADRATIC EQUATIONS, or equations of the *second* degree, are divided into two classes—*pure* and *affected*

220. A PURE QUADRATIC contains only *one* power of the unknown quantity, and that is always the *square*; thus, $2x^2=18$, $ax^2+b=c$, etc. are *pure* quadratics.

Equations of this class are also called *equations of two terms*, because they can always be expressed in two terms— viz. one term containing the square of the unknown quantity, and the other the known quantity.

221. An AFFECTED QUADRATIC contains the first power of the unknown quantity in one term, and the square of that quantity in another term; thus, $x^2+3x=25$, $ax^2+bx=c$, are *affected quadratics*.

PURE QUADRATICS.

222. Ex. 1. Given $4x^2-23=2x^2+27$, to find the value of x.

Analysis. We first transpose and unite the terms, and get Eq. (2); then, dividing by 2, we get Eq. (3); we then extract the square root of both sides, and find $x=\pm 5$.

OPERATION.

$$4x^2-23=2x^2+27 \qquad (1)$$
$$2x^2=50 \qquad (2)$$
$$x^2=25 \qquad (3)$$
$$x=\pm 5 \qquad (4)$$

An *even* root of a positive quantity is always ambiguous (189); hence, a pure quadratic always has *two roots, which are equal numerically, but which have opposite signs.*

2. Given $\frac{x^2}{3}-\frac{x^2}{4}+5=8$, to find x.

Analysis. We first clear the equation of fractions, and get Eq. (2); then transpose and unite terms, and obtain Eq. (3); finally, extracting the square root of both sides, we find $x=\pm 6$.

OPERATION.

$$\frac{x^2}{3}-\frac{x^2}{4}+5=8 \qquad (1)$$
$$4x^2-3x^2+60=96 \qquad (2)$$
$$x^2=36 \qquad (3)$$
$$x=\pm 6$$

3. Given $6x^2-4=212$, to find x. *Ans.* $x=\pm 6$.

4. Given $7x^2+7=3x^2+43$, to find x. *Ans.* $x=\pm 3$.

5. Given $3x^2+5=\frac{3x^2}{4}+41$, to find x. *Ans.* $x=\pm 4$.

6. Given $x^2-14=130-3x^2$, to find x. *Ans.* $x=\pm 6$.

7. Given $6+2x^2=\frac{3x^2}{5}+41$, to find x. *Ans.* $x=\pm 5$.

8. Given $\frac{3x^2+6}{9}+6=40$, to find x. *Ans.* $x=\pm 10$

9. Given $\frac{2x^2+12}{5}=\frac{x^2+20}{3}$, to find x. *Ans.* $x=\pm 8$.

10. Given $5x^2-ac=2x^2$, to find x. *Ans.* $x=\pm\sqrt{\frac{ac}{3}}$.

11. Given $\frac{x-3}{4}=\frac{4}{x+3}$, to find x. *Ans.* $x=\pm 5$.

12. Given $\frac{a}{x+c}=\frac{x-c}{a}$, to find x. *Ans.* $x=\pm\sqrt{a^2+c^2}$.

13. Given $\frac{3x^2-3}{5}=\frac{x^2}{3}+9$, to find x. *Ans.* $x=\pm 6$.

14. Given $\frac{5x^2-8}{4}=x^2+2$, to find x. *Ans.* $x=\pm 4$.

15. Given $(x+4)^2=8x+25$, to find x. *Ans.* $x=\pm 3$.

16. Given $\frac{2x^2+4}{3}-\frac{3x^2-5}{5}=4$, to find x. *Ans.* $x=\pm 5$.

17. Given $(x+5)^2=10x+41$, to find x. *Ans.* $x=\pm 4$.

18. Given $\frac{2}{x^2}+\frac{3}{4}=\frac{35}{36}$, to find x. *Ans.* $x=\pm 3$.

19. Given $3+\frac{x^2}{4}-5=\frac{3x^2}{8}-4$, to find x. *Ans.* $x=\pm 4$.

20. Given $\sqrt{x+3}=\frac{4}{\sqrt{x-3}}$, to find x.

Analysis. We first clear the equation of fractions by multiplying both sides by $\sqrt{x-3}$, and obtain Eq. (2); we then square both sides, which removes the radical sign; then transposing we get $x^2=25$; and extracting the square root of both sides, we find $x=\pm 5$.

OPERATION.

$$\sqrt{x+3}=\frac{4}{\sqrt{x-3}} \qquad (1)$$

$$\sqrt{x^2-9}=4 \qquad (2)$$

$$x^2-9=16 \qquad (3)$$

$$x^2=25 \qquad (4)$$

$$x=\pm 5 \qquad (5)$$

21. Given $\sqrt{5-x}=\dfrac{3}{\sqrt{5+x}}$, to find x. *Ans.* $x=\pm 4$.

22. Given $\sqrt{x-a}=\dfrac{a+d}{\sqrt{x+a}}$, to find x.

Ans. $x=\pm\sqrt{2a^2+2ad+d^2}$.

23. Given $\frac{1}{2}\sqrt{3x^2+16}=2\sqrt{x^2-12}$, to find x.

Ans. $x=\pm 4$.

PROBLEMS.

Ex. 1. There is a certain number to which if you add 4, and then extract the square root of the sum, and from this root subtract 3, the remainder will be 5; what is the number?

Let $x=$ the number.

Then by the conditions, $\sqrt{x+4}-3=5$;

transposing and uniting, $\sqrt{x+4}=8$;

squaring both sides, $x+4=64$,

$x=60$. *Ans.* $x=60$.

2. The product of two numbers is 324, and one of the numbers is 4 times as large as the other; what are the numbers? *Ans.* 9 and 36.

3. If $\frac{1}{4}$ of a certain number be multiplied by $\frac{1}{3}$ of the same, the product will be 108; what is the number?

Ans. 36.

4. There is a certain number such that twice its square increased by 225 is equal to three times its square diminished by 351; what is the number? *Ans.* 24.

5. A merchant bought a piece of muslin for $20.25, and each yard cost as many cents as there were yards in the piece; how many yards were there? *Ans.* 45 yards.

6. What number is that the square of one-half of which is greater than the square of one-third of it by 80?

Ans. 24.

7. The product of two numbers is 1470, and one of the

numbers is only $\frac{5}{6}$ as great as the other; what are the numbers? *Ans.* 42 and 35.

8. The sum of two numbers is 18, and their product is 72; what are the two numbers?

Let $x =$ half the *difference* of the two numbers;
then $9 =$ " *sum* " "

And as *half the sum of any two numbers added to half their difference will give the* GREATER, *and half their difference taken from half their sum will give the* LESS (**159**), it follows that

$9 + x =$ the *greater* number,
$9 - x =$ the *less* "
and $81 - x^2 =$ the product of the two numbers;
therefore, $81 - x^2 = 72$;
transposing, $81 - 72 = x^2$,
or, $x^2 = 81 - 72 = 9$.
Extracting square root, $x = \pm 3$;
then, $9 + x = 9 + 3 = 12$ greater,
$9 - x = 9 - 3 = 6$ less. *Ans.* 12 and 6.

9. The product of two numbers is 60, and their sum is 16; what are the numbers? *Ans.* 6 and 10.

10. The sum of two numbers is 32, and their product is 240; what are the numbers? *Ans.* 12 and 20.

11. The product of two numbers is 384, and the smaller number is $\frac{2}{3}$ of the larger; what are the numbers? *Ans.* 16 and 24.

12. A gentleman's age is such that if from its square you subtract 75, and divide the remainder by 3, the quotient will be 167; what is his age? *Ans.* 24.

13. A gentleman being asked the distance to a certain place, replied, "If you divide the square of the distance by 3, and add 486 to the quotient, the result will be equal to the distance squared;" what was the distance? *Ans.* 27 miles.

14. The number of gallons in a cask of wine is such that

if the square of the number of gallons be divided by 6, and this quotient be increased by 756, the sum will equal $\frac{3}{4}$ of the square of the whole number of gallons; how much does the cask hold? *Ans.* 36 gal.

15. If $\frac{1}{3}$ of the square of a certain number be taken from 210, the remainder will be equal to $\frac{1}{9}$ of the square of the number increased by 14; what is the number? *Ans.* 21.

AFFECTED QUADRATICS.

223. An AFFECTED QUADRATIC EQUATION is one which contains the first power of the *unknown quantity in one term, the square of the unknown quantity in another term, and the known quantity in the third term;* as, $ax^2+bx=c$, or $x^2+4x=12$.

The unknown quantity may *originally* be in several terms, but they may all be reduced to *two*—one containing the first power of the unknown quantity, and the other containing its square.

Equations of this class are also sometimes called *equations of three terms,* because they can always be expressed in three terms—viz., one term containing the first power of the unknown quantity, another term the square of the unknown quantity, and the third term the known quantity.

Thus the equation

$$2x^2-16x=64-2x^2+8x$$

can be reduced by transposition and uniting terms to the form

$$4x^2-24x=64,$$

$$\text{or,}\qquad x^2-6x=16,$$

in which we have the unknown quantity in one term, and the *square* of it in another.

Let us take this equation: $x^2-6x=16$. Suppose we are required to find the value of x. As the equation now stands, we cannot find the square root of the left-hand side, because *it is not a complete square.*

Let us add 9 to both sides. The equality will still be preserved, and we shall have $x^2-6x+9=16+9=25.$

The left-hand member of the equation is now a *complete square,* and we can *extract its square root.* Taking the square root of both sides, we have
$$x-3=\pm 5,$$
and
$$x=3\pm 5=8, \text{ or } -2.$$

To *prove* these values to be correct, we substitute the values of x in the equation $x^2-6x=16$. First, take $x=8$, and we have
$$8^2-6\times 8=16,$$
or,
$$64-48=16,$$
and,
$$16=16.$$

Taking the other value of x (viz. $x=-2$), and substituting, we have
$$(-2)^2-(6\times -2)=16,$$
or,
$$4+12=16,$$
$$16=16.$$

By adding 9 to both sides of the equation, we made the left-hand side a *complete* square. But 9 is the square of 3, and 3 is *half* of 6, which is the coefficient of x; therefore, we added to both sides the *square of half the coefficient of the lowest power of the unknown quantity.* This operation is called *completing the square.*

Hence, to complete the square in an affected quadratic equation we employ the following

RULE.—*Take the square of half the coefficient of the lowest power of the unknown quantity, and add it to both sides of the equation.*

2. Given $x^2-4x=12$, to find the value of x.

OPERATION.

$$x^2-4x=12 \quad (1)$$
$$x^2-4x+4=12+4=16 \quad (2)$$
$$x-2=\pm 4 \quad (3)$$
$$x=2\pm 4=6, \text{ or } -2 \quad (4)$$

Analysis. We first complete the square by taking *half the coefficient* of x (viz. 2), squaring it, making 4, and adding it to both sides of the equation, thus giving Eq. (2). We then extract the square root of both sides, and reducing find the value of $x=6$, or $x=-2$.

Ans. $x=6$, or $x=-2$.

3. Given $x^2+10x=39$, to find x.

Analysis. Here the coefficient of the lowest power is 10; taking *half* of this, and squaring it, making 25, we add it to both sides; then, extracting the square root of both sides and reducing, we have $x=3$, or $x=-13$.

OPERATION.

$$x^2+10x=39 \qquad (1)$$
$$x^2+10x+25=39+25=64 \qquad (2)$$
$$x+5=\pm 8 \qquad (3)$$
$$x=\pm 8-5=3, \text{ or } -13 \qquad (4)$$

Ans. $x=3$, or -13.

4. Given $x^2+ax=b$, to find x.

Analysis. Here the coefficient of x is a. We take $\frac{a}{2}$, and square it, making $\frac{a^2}{4}$, and add it to both sides. We then extract the square root of both members, and transposing and reducing, find the answer. *Ans.* $x=-\frac{a}{2}\pm\sqrt{\frac{a^2}{4}+b}$.

OPERATION.

$$x^2+ax=b \qquad (1)$$
$$x^2+ax+\frac{a^2}{4}=\frac{a^2}{4}+b \qquad (2)$$
$$x+\frac{a}{2}=\pm\sqrt{\frac{a^2}{4}+b} \qquad (3)$$
$$x=-\frac{a}{2}\pm\sqrt{\frac{a^2}{4}+b} \qquad (4)$$

The solution of every affected quadratic equation always gives *two* values for the unknown quantity. These may be both positive or both negative, but usually one will be positive and the other negative.

5. Given $x^2-8x=9$, to find x. *Ans.* $x=9$, or -1.
6. Given $x^2-10x=24$, to find x. *Ans.* $x=12$, or -2.
7. Given $x^2-2x=24$, to find x. *Ans.* $x=6$, or -4.
8. Given $x^2+6x=27$, to find x. *Ans.* $x=3$, or -9.
9. Given $x^2-12x=28$, to find x. *Ans.* $x=14$, or -2.
10 Given $x^2-6x=27$, to find x. *Ans.* $x=9$, or -3
11. Given $x^2+2x=35$, to find x. *Ans.* $x=5$, or -7.
12. Given $x^2+4x=45$, to find x. *Ans.* $x=5$, or -9.
13. Given $x^2+8x=20$, to find x. *Ans.* $x=2$, or -10.
14. Given $x^2+14x=32$, to find x. *Ans.* $x=2$, or -16.

15. Given $x^2+3x=18$, to find x.

Analysis. Here the coefficient of x is 3, an *odd* number. We take *half* of 3, making $\frac{3}{2}$, and square it, giving $\frac{9}{4}$, and add it to both sides. Then, extracting the square root of both sides and reducing, we find $x=3$, or $x=-6$.

Ans. $x=3$, or -6.

OPERATION.

$$\begin{aligned} x^2+3x&=18 && (1)\\ x^2+3x+\tfrac{9}{4}&=\tfrac{9}{4}+18=\tfrac{81}{4} && (2)\\ x+\tfrac{3}{2}&=\pm\tfrac{9}{2} && (3)\\ x&=-\tfrac{3}{2}\pm\tfrac{9}{2}\\ x&=\tfrac{6}{2}=3\\ x&=-\tfrac{12}{2}=-6 \end{aligned}$$

16. Given $x^2-5x=6$, to find x. *Ans.* $x=6$, or -1.

17. Given $x^2+7x=18$, to find x. *Ans.* $x=2$, or -9.

18. Given $x^2-9x=10$, to find x. *Ans.* $x=10$, or -1.

The *reason* underlying the operation of completing the square will be readily understood by examining the process of squaring any binomial.

Suppose we have the equation $x^2+2ax=d$.

We know that the square of $(x+a)$ is $x^2+2ax+a^2$, and by comparing this square with the left-hand member of our equation above, we see that the first member of our equation lacks the term (a^2) of being a *complete* square. But a^2 is the square of (a), and (a) is *half the coefficient of the first or lowest power* of x. If then we add a^2 to the left-hand side of the equation, this side will become a *complete square.* In order to preserve the equality of the sides, we must add a^2 also to the right-hand side. Our equation now stands,

$$x^2+2ax+a^2=a^2+d.$$

Extracting the square root of both sides, we have

$$x+a=\pm\sqrt{a^2+d},$$

and $$x=-a\pm\sqrt{a^2+d}.$$

19. Given $3x^2+2x-4=x^2-6x+60$, to find x.

Analysis. We first transpose and unite the terms, bringing the unknown quantities on one side and the known on the other. Then we divide each side by 2, and get Eq. (3). We then complete the square, and extract the square root of both sides, and reducing, find $x=4$, or $x=-8$.

OPERATION.

$$\begin{aligned} 3x^2+2x-4&=x^2-6x+60 && (1)\\ 2x^2+8x&=64 && (2)\\ x^2+4x&=32 && (3)\\ x^2+4x+4&=4+32=36 && (4)\\ x+2&=\pm6 && (5)\\ x&=-2\pm6=4,\text{ or }-8 && (6) \end{aligned}$$

Ans. $x=4$, or $x=-8$.

20. Given $5x^2+240-6x=2x^2+285$, to find x.

Ans. $x=5$, or -3.

21. Given $4x^2-10x-200=x^2+2x+295$, to find x.

Ans. $x=15$, or -11.

22. Given $2x^2-10=6x+10$, to find x.

Ans. $x=5$, or -2.

23. Given $3x^2+50=12x+86$, to find x.

Ans. $x=6$, or -2.

24. Given $24-x^2+6x=2x-8$, to find x.

OPERATION.

$$24-x^2+6x=2x-8 \qquad (1)$$
$$-x^2+4x=-32 \qquad (2)$$
$$x^2-4x=32 \qquad (3)$$
$$x^2-4x+4=4+32=36 \qquad (4)$$
$$x-2=\pm 6 \qquad (5)$$
$$x=2\pm 6=8, \text{ or } -4 \qquad (6)$$

Analysis. We first transpose and unite terms, and get Eq. (2). We then change the signs of all the terms by dividing the equation through by -1, so that the *highest* power of the unknown quantity may have the sign +. We then complete the square, and extract the square root, and reducing, find $x=8$, or $x=-4$.

NOTE.—The necessity of changing the signs, as in the last example, is owing to the fact that the square root of a *negative* quantity is *impossible* (189). Whenever, therefore, $-x^2$ occurs in an equation, it is impossible to find its square root; but by changing the signs of *all* the terms, x^2 then becomes *positive*, and its square root may be extracted.

25. Given $10x-x^2=8x-15$, to find x.

Ans. $x=5$, or -3.

26. Given $27-x^2+6x=48-4x$, to find x.

Ans. $x=7$, or 3.

27. Given $4x+30-x^2=25$, to find x. *Ans.* $x=5$, or -1.

28. Given $22+8x-x^2=5x+4$, to find x.

Ans. $x=6$, or -3.

29. Given $32+3x-x^2=5x-88$, to find x.

Ans. $x=10$, or -12.

30. Given $3x^2-2x=21$, to find x.

Analysis. We first divide the whole equation by 3, to free the highest power of the unknown quantity from coefficients; we then complete the square by taking *half* the coefficient of x (viz. $\frac{1}{3}$), squaring it, and adding it to both sides; then we extract the square root, and reducing, we get $x=3$, or $x=-\frac{7}{3}$.

OPERATION.

$$3x^2-2x=21 \tag{1}$$

$$x^2-\frac{2x}{3}=7 \tag{2}$$

$$x^2-\frac{2x}{3}+\frac{1}{9}=\frac{1}{9}+7=\frac{64}{9} \tag{3}$$

$$x-\frac{1}{3}=\pm\frac{8}{3} \tag{4}$$

$$x=\frac{1}{3}\pm\frac{8}{3}=\frac{9}{3}, \text{ or } -\frac{7}{3} \tag{5}$$

$$x=3, \text{ or } -\frac{7}{3} \tag{6}$$

31. Given $2x^2-3x=2$, to find x. *Ans.* $x=2$, or $-\frac{1}{2}$.

32. Given $3x^2+2x-10=23$, to find x.
Ans. $x=3$, or $-\frac{11}{3}$.

33. Given $2x^2+5x+4=22$, to find x. *Ans.* $x=2$, or $-\frac{9}{2}$.

34. Given $3x-\frac{16-x}{x}=9$, to find x.

Analysis. We first clear the equation of fractions, and get Eq. (2). Transposing and uniting, we have Eq. (3). Dividing by 3, the coefficient of x^2, we get Eq. (4); as $\frac{8}{3}$ is the coefficient of x, we add $\frac{16}{9}$, the square of *half* of $\frac{8}{3}$, to both sides, and obtain Eq. (5). Extracting the square root of both sides, and reducing, we find $x=3$, or $x=-\frac{4}{3}$.

OPERATION.

$$3x-\frac{16-x}{x}=9 \tag{1}$$

$$3x^2-16+x=9x \tag{2}$$

$$3x^2-8x=16 \tag{3}$$

$$x^2-\frac{8x}{3}=\frac{16}{3} \tag{4}$$

$$x^2-\frac{8x}{3}+\frac{16}{9}=\frac{16}{9}+\frac{16}{3}=\frac{64}{9} \tag{5}$$

$$x-\frac{4}{3}=\pm\frac{8}{3} \tag{6}$$

$$x=\frac{4}{3}\pm\frac{8}{3}=\frac{12}{3}, \text{ or } -\frac{4}{3} \tag{7}$$

$$x=4, \text{ or } -\frac{4}{3}$$

224. All that we have said on the subject of solving affected quadratics may be summed up in the following general

RULE.—*I. Clear the equation of fractions.*

II. Transpose and unite the terms, so that the unknown quantities shall stand on one side and the known quantities on the other.

III. Complete the square by adding to both sides of the equation the square of half the coefficient of the first power of the unknown quantity.

IV. Extract the square root of both sides, and reduce the resulting equation.

NOTE.—It must be borne in mind that the *highest* power of the unknown quantity must always stand first—must have the positive sign and be free from coefficients before we attempt to complete the square by the rule just given.

EXAMPLES.

1. Given $x^2+6x=40$, to find x. *Ans.* $x=4$, or -10.
2. Given $x^2-4x+10=15$, to find x. *Ans.* $x=5$, or -1.
3. Given $x^2+8x+6=26$, to find x. *Ans.* $x=2$, or -10.
4. Given $x^2-2x+5=85$, to find x. *Ans.* $x=10$, or -8.
5. Given $x^2+10x+7=18$, to find x. *Ans.* $x=1$, or -11.
6. Given $x^2+12x+10=55$, to find x. *Ans.* $x=3$, or -15.
7. Given $x^2+3x+7=25$, to find x. *Ans.* $x=3$, or -6.
8. Given $x^2+7x-5=39$, to find x. *Ans.* $x=4$, or -11.
9. Given $x^2-6x+10=2$, to find x. *Ans.* $x=4$, or 2.
10. Given $x^2+5x+4=18$, to find x. *Ans.* $x=2$, or -7.
11. Given $x^2-8x+50=35$, to find x. *Ans.* $x=5$, or 3.
12. Given $3x^2+12x+6=42$, to find x. *Ans.* $x=2$, or -6.
13. Given $2x^2+8x-10=80$, to find x. *Ans.* $x=5$, or -9.
14. Given $5x^2+20x=135-10x$, to find x.
    Ans. $x=3$, or -9.
15. Given $3x^2-12x-20=16$, to find x.
    Ans. $x=6$, or -2.
16. Given $4x^2-20x+24=x^2-2x+45$, to find x.
    Ans. $x=7$, or -1.

17. Given $\frac{x^2}{3}+2x+10=31-4x$, to find x.
Ans. $x=3$, or -21.

18. Given $21+\frac{x^2-6}{5}=4x+3$, to find x.
Ans. $x=14$, or 6.

19. Given $x-\frac{12-x}{x}=5$, to find x. *Ans.* $x=6$, or -2.

20. Given $\frac{24}{x}=\frac{24}{x+2}+2$, to find x. *Ans.* $x=4$, or -6.

21. Given $x^2-4x+10=\frac{x^2+8x+10}{5}$, to find x.
Ans. $x=5$, or 2.

22. Given $\frac{x+2}{x}=\frac{2x+4}{x+4}$, to find x *Ans.* $x=4$, or -2.

23. Given $x+\frac{25}{x-1}=2x-1$, to find x. *Ans.* $x=6$, or -4.

SECOND METHOD OF COMPLETING THE SQUARE.

225. By the method already given any affected quadratic may be solved; but frequently when there are fractions the work becomes complicated and tedious. There is another method by which fractions can be avoided to a considerable extent.

Suppose we have the equation

$$x^2+3x=18.$$

Completing the square by the rule previously given, we have

$$x^2+3x+\tfrac{9}{4}=\tfrac{9}{4}+\tfrac{72}{4}=\tfrac{81}{4}.$$

Each side of this equation is a *complete* square, and therefore its square root can be extracted and the value of x can be found.

Suppose, however, that we multiply both sides of this completed square by 4; we shall then have

$$4x^2+12x+9=81.$$

A little examination will show us that both sides of this last equation are *complete squares*, but with this advantage—

that there are no *fractions.* We derived this last equation from the *original* one simply by multiplying the *whole equation by four times the coefficient of the highest power of the unknown quantity, and adding to both sides the square of the coefficient of the lowest power.*

Ex. 1. Given $x^2-5x=14$, to find x.

Analysis. We first complete the square by multiplying the whole equation by 4, and adding to both sides the square of 5, the coefficient of the lowest power of x. Then, extracting the square root and reducing, we find $x=7$, or $x=-2$.

OPERATION.

$$\begin{array}{cr} x^2-5x=14 & (1) \\ 4x^2-20x+25=25+56=81 & (2) \\ 2x-5=\pm 9 & (3) \\ 2x=5\pm 9 & (4) \\ 2x=14,\text{ or }-4 & (5) \\ x=7,\text{ or }-2 & (6) \end{array}$$

Ans. $x=7$, or -2

2. Given $3x^2-2x=21$, to find x.

Analysis. Here the coefficient of x^2 is 3; hence, we multiply the whole equation by 4 times 3, or 12, and add to both sides the square of 2, the coefficient of x. Then, extracting the square root and reducing, we have $x=3$, or $x=-\frac{7}{3}$.

OPERATION.

$$\begin{array}{cr} 3x^2-2x=21 & (1) \\ 36x^2-24x+4=4+252=256 & (2) \\ 6x-2=\pm 16 & (3) \\ 6x=2\pm 16=18,\text{ or }-14 & (4) \\ x=3,\text{ or }-\frac{14}{6} & (5) \\ x=3,\text{ or }-\frac{7}{3}. & (6) \end{array}$$

Ans. $x=3$, or $-\frac{7}{3}$.

3. Given $5x^2+7x=66$, to find x.

Analysis. Here the coefficient of x^2 is 5; hence, we multiply the whole equation by 4 times 5, or 20; and add to both sides the square of 7, the coefficient of x. Then, extracting the square root and reducing, we get $x=3$, or $x=-4\frac{4}{10}$.

OPERATION.

$$\begin{array}{cr} 5x^2+7x=66 & (1) \\ 100x^2+140x+49=49+1320=1369 & (2) \\ 10x+7=\pm 37 & (3) \\ 10x=-7\pm 37=30,\text{ or }-44 & (4) \\ x=3,\text{ or }-4\frac{4}{10} & (5) \end{array}$$

Ans. $x=3$, or $-4\frac{4}{10}$.

Hence, for completing the square we have a second

RULE.—*Multiply the whole equation by four times* *the coefficient of the highest power of the unknown quantity, and add to both sides the square of the coefficient of the lowest power.*

The student must remember *always* to reduce the equation to that form in which the *highest power* of the unknown quantity shall be the first term, and have the sign *plus*, before attempting to complete the square by this rule.

NOTE.—Students are sometimes at a loss to know when to employ the first rule, and when the second. One simple suggestion will relieve them of all trouble. Whenever the coefficient of the highest power is 1, and the coefficient of the lowest power is an *even* number, use the *first* rule; in all other cases use the *second*.

EXAMPLES.

4. Given $x^2-10x+12=23$, to find x. *Ans.* $x=11$, or -1.

5. Given $2x^2+3x+6=50$, to find x. *Ans.* $x=4$, or $-5\frac{1}{2}$.

6. Given $2x^2-8x+12=22$, to find x. *Ans.* $x=5$, or -1.

7. Given $3x^2-2x+10=75$, to find x. *Ans.* $x=5$, or $-4\frac{1}{3}$.

8. Given $4x^2-8x+32=64$, to find x. *Ans.* $x=4$, or -2.

9. Given $3x^2+5x-20=22$, to find x. *Ans.* $x=3$, or $-4\frac{2}{3}$.

10. Given $x^2-6x+15=31$, to find x. *Ans.* $x=8$, or -2.

11. Given $4x^2+x-8=10$, to find x. *Ans.* $x=2$, or $-2\frac{1}{4}$.

12. Given $2x-\frac{12-x}{x}=11$, to find x. *Ans.* $x=6$, or -1.

13. Given $\frac{x}{3}+\frac{12}{x-3}=6$, to find x. *Ans.* $x=15$, or 6.

14. Given $\frac{36}{x}=\frac{36}{x+3}+2$, to find x. *Ans.* $x=6$, or -9.

15. Given $\frac{42}{x}+1=\frac{42}{x-1}$, to find x. *Ans.* $x=7$, or -6.

16. Given $3x-\frac{12-x}{x}=10$, to find x. *Ans.* $x=4$, or -1.

17. Given $\frac{x}{2}+\frac{10}{x+3}=3$, to find x. *Ans.* $x=2$, or 1.

18. Given $\frac{10+x}{x}+\frac{10-x}{5}=4$, to find x.
Ans. $x=5$, or -10.

19. Given $2x^2-3ax+b=cd$, to find x.
Ans. $x=\frac{3a}{4}\pm\frac{\sqrt{9a^2+8cd-8b}}{4}$

PROBLEMS PRODUCING AFFECTED QUADRATIC EQUATIONS.

1. Divide 40 into two such parts that their product shall be 375.

Let $x=$ one part,
then, $40-x=$ the other part,
$40x-x^2=$ the product.
Then, by the conditions, $40x-x^2=375$,
or, $-x^2+40x=375$;
dividing by -1, to change the signs, $x^2-40x=-375$;
completing the sq. by 1st rule, $x^2-40x+400=400-375=25$;
extracting square root, $x-20=\pm 5$,
$x=20\pm 5=25$, or 15,
$40-x=15$, or 25.

The parts therefore are 25 and 15.

If, at the outset, we had said, Let $x=$ the *greater* part, and $40-x=$ the *less*, then it is obvious that the value of x would have been 25, and *only* 25; and the *less* part would necessarily have been 15. Although in the solution of the question x has *two* values, yet by limiting x to represent the *greater* part, this limitation of its value makes it, therefore, represent only *one* quantity—viz. 25.

On the contrary, if we had said, Let $x=$ the *less* part, then the value of x would necessarily have been 15, and *only* 15.

But when, as in our question, we said, Let $x=$ one part, we left the matter *indefinite* and hence x may properly represent either the *greater* or the *less*, and therefore may be either 25 or 15. Hence, while the solution of any affected quadratic equation uniformly gives *two* values, yet it is not always the case that both of these values will satisfy the problem as it is literally stated.

In succeeding problems we shall give only the positive values of the unknown quantity.

2. Divide 36 into two parts, so that their product shall be 320. *Ans.* 20 and 16.

3. The sum of two numbers is 42, and the sum of their squares is 884; what are the numbers? *Ans.* 22 and 20.

4. A laborer was employed to dig two ditches, one of which was 10 yards longer than the other. Each ditch cost as many cents per yard as there were yards in the ditch, and the whole amount paid out was $50.50; what was the length of each ditch? *Ans.* 45 and 55 yards.

5. Divide 24 into two such parts that the square of the greater shall be equal to 4 times the square of the less. *Ans.* 16 and 8.

6. A man bought a certain number of sheep for $48. If he had bought 2 more for the same money, they would have cost him $2 less per head; how many did he buy?

Let $x=$ No. he bought,

$\frac{48}{x}=$ cost of each,

$\frac{48}{x+2}=$ cost of each if he had bought 2 more.

By the conditions, $\frac{48}{x}=\frac{48}{x+2}+2,$

clearing of fractions, $48x+96=48x+2x^2+4x;$

transposing & uniting, $-2x^2-4x=-96;$

dividing by -1, to change the signs, $2x^2+4x=96;$

dividing by 2, $x^2+2x=48;$

completing the square, $x^2+2x+1=1+48=49;$

extracting the square root, $x+1=\pm 7;$

transposing and uniting, $x=-1\pm 7=6$, or -8.

The *proof* of this is obvious. If he bought 6 sheep for $48, he evidently paid \$8 apiece for them. If he had bought 2 more (*i. e.* 8), he would have paid only \$6 apiece for them—*two* dollars less than in the first instance.

The value -8 for x is not applicable to things like sheep.

7. A merchant bought a certain number of hats for \$18; if he had bought 3 hats less, he would have paid \$1 more apiece for them; how many did he buy? *Ans.* 9 hats.

8. A rectangular garden contains 4320 square yards, and it is 12 yards longer than it is broad; what is its length and breadth? *Ans.* 72 and 60 yards.

9. The difference of two numbers is 4, and their product is 252; what are the numbers? *Ans.* 18 and 14.

10. The carpet on a certain floor contains 270 square feet, and its breadth is 3 feet less than its length; what are its dimensions? *Ans.* 18 and 15 feet.

11. The difference between the ages of two brothers is 6 years, and the sum of the squares of their ages is 2196; what is the age of each? *Ans.* 30 and 36.

12. Two numbers differ by 2, and if 40 be added to twice the square of the greater, it will be equal to 3 times the square of the less; what are the numbers? *Ans.* 12 and 14.

13. A man went on a journey of 60 miles in a certain time, but if he had travelled 2 miles faster per hour he could have accomplished his journey in 5 hours less time; what was his speed per hour? *Ans.* 4 miles.

14. Twice the square of a certain number, diminished by 350, is equal to 5 times the number itself, increased by 25; what is the number? *Ans.* 15.

15. The difference of two numbers is 5, and the sum of their squares is equal to 3 times their product, diminished by 125; what are the numbers? *Ans.* 10 and 15.

16. A merchant bought a certain number of yards of

broadcloth for $60. If he had bought 5 yards more for the same money, he would have paid $2 less per yard; how many yards did he buy? *Ans.* 10 yards.

17. The sum of two numbers is 24, and 5 times their product is equal to twice the sum of their squares; what are the numbers? *Ans.* 8 and 16.

18. A certain number of persons agreed to raise 80 dollars, each subscribing an equal sum. Two of them having failed to comply, the amount was paid by the others, each of whom paid 2 dollars more than would otherwise have been his share; how many persons were there? *Ans.* 10.

19. If a certain number be subtracted from 32, and the remainder be multiplied by the number itself, the product will be 252; what is the number? *Ans.* 18 or 14.

20. What number is that which being added to 12, and the sum multiplied by the number itself, the product will be 189? *Ans.* 9.

21. Divide 60 into two such parts that the square of the greater diminished by 64 shall be equal to 3 times the square of the less increased by 336. *Ans.* 40 and 20.

22. Two numbers differ by 6, and one-third of the square of the larger number is equal to the square of the smaller diminished by 36; what are the two numbers?

Ans. 12 and 18.

23. A certain orchard contains 594 trees, and there are 15 more trees in a row than there are rows; how many rows and how many trees are there? *Ans.* 18 rows; 33 trees.

24. A merchant bought a certain number of pieces of silk for $320. If he had bought 4 more pieces for the same money, he would have paid $4 less for each piece; how many pieces did he buy? *Ans.* 16 pieces.

25. The united ages of a gentleman and his wife are 64

years, and the sum of the squares of their separate ages is 2056; what are their ages? *Ans.* 30 and 34.

26. A merchant bought 2 pieces of silk velvet, paying \$210 for each piece. One piece measured 7 yards longer than the other, and cost \$1 less per yard than the other: how many yards were in each piece?

Ans. 35 yards; 42 yards.

27. The product of two numbers is 72, and if the greater be diminished by 3, and the less increased by 2, the product of the sum and remainder will be 60; what are the numbers?

Let $x =$ the greater,

$\frac{72}{x} =$ the less,

$x - 3 =$ the greater diminished by 3

$\frac{72}{x} + 2 =$ the less increased by 2.

By the conditions, $(x-3)\left(\frac{72}{x}+2\right) = 60;$

expanding, $66 + 2x - \frac{216}{x} = 60;$

clearing of fractions, $66x + 2x^2 - 216 = 60x;$

transposing and uniting, $2x^2 + 6x = 216;$

$x^2 + 3x = 108;$

completing the square, $4x^2 + 12x + 9 = 9 + 432 = 441;$

extracting the sq. root, $2x + 3 = \pm 21,$

$2x = -3 \pm 21 = 18, \text{ or } -24,$

$x = 9, \text{ or } -12,$

$\frac{72}{x} = 8, \text{ or } -6.$

28. The product of two numbers is 144, and if the greater be increased by 2 and the less be diminished by 1, the product of the resulting numbers will still be 144; what are the numbers? *Ans.* 16 and 9.

29. Two numbers are to each other in the ratio of 2 to 3, and if 4 be added to the less and subtracted from the

greater, the product of the sum and remainder will be 400; what are the numbers? *Ans.* 16 and 24.

EQUATIONS IN THE QUADRATIC FORM.

226. Any equation which contains only two powers of the unknown quantity, the *index* of one of which is *double* that of the other, may be solved by the rules already given for *affected* quadratics. We present a few examples:

Ex. 1. Given $x^{\frac{1}{2}}+x^{\frac{1}{4}}=12$, to find the value of x.

Analysis. We first complete the square by taking *half the coefficient* of $x^{\frac{1}{4}}$ (viz. $\frac{1}{2}$), squaring it, making $\frac{1}{4}$, and adding it to both sides of the equation, thus giving Eq. (2). We then extract the square root of both sides, and reducing, find the value of $x^{\frac{1}{4}}=3$, or -4. Then we raise both sides to the *fourth* power, and find $x=81$, or $x=256$.

OPERATION.

$$x^{\frac{1}{2}}+x^{\frac{1}{4}}=12 \qquad (1)$$
$$x^{\frac{1}{2}}+x^{\frac{1}{4}}+\tfrac{1}{4}=\tfrac{48}{4}+\tfrac{1}{4}=\tfrac{49}{4} \qquad (2)$$
$$x^{\frac{1}{4}}+\tfrac{1}{2}=\pm\tfrac{7}{2} \qquad (3)$$
$$x^{\frac{1}{4}}=3, \text{ or } -4 \qquad (4)$$
$$x=81, \text{ or } 256 \qquad (5)$$

Proof. Substituting 81, we have $9+3=12$,

$12=12$;

substituting 256, we have $16-4=12$,

$12=12$.

2. Given $x+3\sqrt{x}=28$, to find x. *Ans.* $x=16$, or 49.

3. Given $2x+4\sqrt{x}=16$, to find x. *Ans.* $x=4$, or 16.

4. Given $x^{\frac{1}{2}}+4(x)^{\frac{1}{4}}=12$, to find x. *Ans.* $x=16$, or 1296.

5. Given $x^{\frac{2}{3}}+x^{\frac{1}{3}}=6$, to find x.

Analysis. We first complete the square by taking half the coefficient of $x^{\frac{1}{3}}$ (viz. $\frac{1}{2}$), squaring it, making $\frac{1}{4}$, and adding it to both sides of the equation. We then extract the square root of both sides, and reducing, obtain the value of $x^{\frac{1}{3}}=2$, or -3 We then raise both sides to the *third* power, and find $x=8$, or $x=-27$.

OPERATION.

$$x^{\frac{2}{3}}+x^{\frac{1}{3}}=6 \qquad (1)$$
$$x^{\frac{2}{3}}+x^{\frac{1}{3}}+\tfrac{1}{4}=\tfrac{24}{4}+\tfrac{1}{4}=\tfrac{25}{4} \qquad (2)$$
$$x^{\frac{1}{3}}+\tfrac{1}{2}=\pm\tfrac{5}{2} \qquad (3)$$
$$x^{\frac{1}{3}}=-\tfrac{1}{2}\pm\tfrac{5}{2}$$
$$x^{\frac{1}{3}}=2, \text{ or } -3$$
$$x=8, \text{ or } -27$$

6. Given $x^{\frac{2}{3}}-2(x)^{\frac{1}{3}}=8$, to find x. *Ans.* $x=64$, or -8.

7. Given $x^{\frac{1}{3}}+x^{\frac{1}{6}}=6$, to find x. *Ans.* $x=64$, or 729.

8. Given $x^4-3x^2=54$, to find x.

Analysis. We first complete the square by taking half the coefficient of x^2 (viz. $\frac{3}{2}$), squaring it, making $\frac{9}{4}$, and adding it to both sides of the equation. We then extract the square root of both sides, and reducing, find the value of $x^2=9$, or -6. Then, taking the square root of both sides, we find the value of $x=\pm 3$, or $\pm\sqrt{-6}$.

OPERATION.

$$x^4-3x^2=54$$
$$x^4-3x^2+\tfrac{9}{4}=\tfrac{9}{4}+\tfrac{216}{4}=\tfrac{225}{4}$$
$$x^2-\tfrac{3}{2}=\pm\tfrac{15}{2}$$
$$x^2=\tfrac{3}{2}\pm\tfrac{15}{2}=9,\text{ or }-6.$$
$$x=\pm 3,\text{ or }\pm\sqrt{-6}$$

9. Given $x^4-2x^2=8$, to find x. *Ans.* $x=\pm 2$, or $\pm\sqrt{-2}$.

10. Given $2x^4-8x^2=90$, to find x.
Ans. $x=\pm 3$, or $\pm\sqrt{-5}$.

11. Given $x^6-10x^3=459$, to find x.

Analysis. We complete the square by taking half the coefficient of x^3 (viz. 5), squaring it, making 25, and adding it to both sides of the equation. We then extract the square root of both sides, and reducing, find the value of $x^3=27$, or -17. Then, extracting the *cube* root of both sides of the equation, we find $x=3$, or $\sqrt{-17}$.

OPERATION.

$$x^6-10x^3=459$$
$$x^6-10x^3+25=25+459=484$$
$$x^3-5=\pm 22$$
$$x^3=5\pm 22=27,\text{ or }-17$$
$$x=3,\text{ or }\sqrt[3]{-17}$$

12. Given $x^6-4x^3=32$, to find x. *Ans.* $x=2$, or $\sqrt[3]{-4}$.

13. Given $3x^6+4x^3=7$, to find x. *Ans.* $x=1$, or $\sqrt[3]{-\frac{7}{3}}$.

14. Given $(10+x)-(10+x)^{\frac{1}{2}}=12$, to find x.

Analysis. We complete the square by taking half the coefficient of $(10+x)^{\frac{1}{2}}$ (viz. $\frac{1}{2}$), squaring it, making $\frac{1}{4}$, and adding it to both sides of the equation. We then extract the square root of both sides, and reducing, find the value

OPERATION.

$$(10+x)-(10+x)^{\frac{1}{2}}=12$$
$$(10+x)-(10+x)^{\frac{1}{2}}+\tfrac{1}{4}=\tfrac{1}{4}+\tfrac{48}{4}=\tfrac{49}{4}$$
$$(10+x)^{\frac{1}{2}}-\tfrac{1}{2}=\pm\tfrac{7}{2}$$
$$(10+x)^{\frac{1}{2}}=\tfrac{1}{2}\pm\tfrac{7}{2}=4,\text{ or }3$$
$$10+x=16,\text{ or }9$$
$$x=6,\text{ or }-1$$

of [illegible], taking the root again and reducing, we find $x = 4$, $y = 2$.

QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

297. Equations in the form of quadratics containing two unknown quantities cannot be solved by any one specific rule, yet the methods presented in the preceding pages will very much assist in their solution. Much is to be left to the skill and ingenuity of the student.

1. Given $\left\{\begin{array}{l} x^2 + y^2 = 29 \\ 2x + y = 12 \end{array}\right\}$, to find x and y.

From the second equation, $y = 12 - 2x$;
then $y^2 = 144 - 48x + 4x^2$.

Substituting this value of y^2 in the first equation, we have

$$x^2 + 144 - 48x + 4x^2 = 29;$$

transposing and uniting, $5x^2 - 48x = -115$;
completing the square, $100x^2 - 960x + 2304 = 2304 - 2300 = 4$,
extracting the square root, $10x - 48 = \pm 2$,

$$10x = 48 \pm 2,$$
$$10x = 50, \text{ or } 46,$$
$$x = 5, \text{ or } 4\tfrac{6}{10},$$
$$y = 12 - 2x = 12 - 10, \text{ or } 12 - 9\tfrac{2}{10},$$
$$y = 2, \text{ or } 2\tfrac{8}{10}.$$

The solution gives *two* values for x, and also two for y. We must be careful to associate the proper values for the two quantities. In the example just solved, if $x = 5$, then $y = 2$; if $x = 4\tfrac{6}{10}$, then $y = 2\tfrac{8}{10}$.

It will be observed that in this example one of the equations is in the form of a quadratic, and the other is a simple equation; and we *found an expression for the value of one of the unknown quantities in the simple equation, and substituted this value for the unknown quantity itself in the other equation.* This process gave a quadratic equation containing only one unknown quantity, which was solved according to the methods given in the preceding pages.

2. Given $\left\{\begin{array}{l} x^2+y^2=25 \\ x+y=7 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=4, \text{ or } 3, \\ y=3, \text{ or } 4. \end{array}\right.$

3. Given $\left\{\begin{array}{l} x^2+y^2=52 \\ x+y=10 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=6, \text{ or } 4, \\ y=4, \text{ or } 6. \end{array}\right.$

4. Given $\left\{\begin{array}{l} 2x^2+xy=30 \\ y-x=1 \end{array}\right\}$, to find x and y.
Ans. $\left\{\begin{array}{l} x=3, \text{ or } -3\frac{1}{3}, \\ y=4, \text{ or } -2\frac{1}{3}. \end{array}\right.$

5. Given $\left\{\begin{array}{l} xy+y^2=36 \\ x=y+1 \end{array}\right\}$, to find x and y.
Ans. $\left\{\begin{array}{l} x=5, \text{ or } -3\frac{1}{2}, \\ y=4, \text{ or } -4\frac{1}{2}. \end{array}\right.$

6. Given $\left\{\begin{array}{l} x^2+xy+y^2=19 \\ x=y-1 \end{array}\right\}$, to find x and y.
Ans. $\left\{\begin{array}{l} x=2, \text{ or } -3, \\ y=3, \text{ or } -2. \end{array}\right.$

7. Given $\left\{\begin{array}{l} 3x^2+y^2=228 \\ x-y=2 \end{array}\right\}$, to find x and y.
Ans. $\left\{\begin{array}{l} x=8, \text{ or } -7, \\ y=6, \text{ or } -9. \end{array}\right.$

8. Given $\left\{\begin{array}{l} x^2+y^2=41 \\ xy=20 \end{array}\right\}$, to find x and y.

From the second equation, $x=\frac{20}{y}$;

then, $x^2=\frac{400}{y^2}$.

Substituting this value of x^2 in the first equation, we have

$$\frac{400}{y^2}+y^2=41;$$

clearing of fractions, $400+y^4=41y^2$;

transposing, $y^4-41y^2=-400$;

completing the square, $4y^4-164y^2+1681=1681-1600=81$;

extracting the sq. root, $2y^2-41=\pm 9$,

$$2y^2=41\pm 9=50, \text{ or } 32,$$

$$y^2=25, \text{ or } 16,$$

$$y=\pm 5, \text{ or } \pm 4,$$

$$x=\frac{20}{y};$$

therefore, $x=\pm 4, \text{ or } \pm 5.$

Ans. $\left\{\begin{array}{l} x=\pm 4, \text{ or } \pm 5, \\ y=\pm 5, \text{ or } \pm 4. \end{array}\right.$

The student will notice the fact that in this example both of the given equations come under the quadratic form (*i. e.* they are both of the *second* degree); and in the progress of the work the process introduces an equation of the *fourth* degree, which, however, is solved exactly in the same manner as an affected quadratic by completing the square and then reducing the resulting equation.

9. Given $\left\{\begin{array}{r} x^2+y^2=52 \\ xy=24 \end{array}\right\}$, to find x and y.

Ans. $\left\{\begin{array}{l} x=\pm 6, \text{ or } \pm 4, \\ y=\pm 4, \text{ or } \pm 6. \end{array}\right.$

10. Given $\left\{\begin{array}{c} x^2+y^2=45 \\ \dfrac{x}{y}=2 \end{array}\right\}$, to find x and y.

Ans. $\left\{\begin{array}{l} x=6, \text{ or } -6, \\ y=3, \text{ or } -3. \end{array}\right.$

11. Given $\left\{\begin{array}{c} \dfrac{8x+2y}{6}=3 \\ 4xy=48 \end{array}\right\}$, to find x and y.

Ans. $\left\{\begin{array}{l} x=2, \text{ or } 4, \\ y=6, \text{ or } 3. \end{array}\right.$

12. Given $\left\{\begin{array}{c} x^2-xy+y^2=19 \\ x-y=2 \end{array}\right\}$, to find x and y.

Ans. $\left\{\begin{array}{l} x=5, \text{ or } -3, \\ y=3, \text{ or } -5. \end{array}\right.$

13. Given $\left\{\begin{array}{c} x+y=6 \\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{3}{4} \end{array}\right\}$, to find x and y. Ans. $\left\{\begin{array}{l} x=4, \text{ or } 2, \\ y=2, \text{ or } 4. \end{array}\right.$

14. Given $\left\{\begin{array}{c} x^2+xy+y^2=37 \\ x-y=1 \end{array}\right\}$, to find x and y.

Ans. $\left\{\begin{array}{l} x=4, \text{ or } -3, \\ y=3, \text{ or } -4. \end{array}\right.$

15. Given $\left\{\begin{array}{c} x^2-y^2=20 \\ x+y=10 \end{array}\right\}$, to find x and y.

OPERATION.

$$x^2-y^2=20, \qquad (1)$$
$$x+y=10. \qquad (2)$$

Dividing the first equation by the second, we have

$$x-y=2; \qquad (3)$$

adding Eq. (2) to (3), we have $2x=12$, (4)

$$x=6,$$
$$y=10-x,$$
$$y=4.$$

Ans. $\left\{\begin{array}{l} x=6, \\ y=4. \end{array}\right.$

16. Given $\left\{\begin{array}{c} x^2-y^2=64 \\ x+y=16 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=10, \\ y=6. \end{array}\right.$

17. Given $\left\{\begin{array}{c} x^2-y^2=72 \\ x-y=6 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x=9, \\ y=3 \end{array}\right.$

18 Given $\left\{\begin{array}{l} x^4-y^4=240 \\ x^2+y^2=20 \end{array}\right\}$, to find x and y.

OPERATION.

$$x^4-y^4=240, \tag{1}$$

$$x^2+y^2=20. \tag{2}$$

Dividing Eq. (1) by Eq. (2), we have

$$x^2-y^2=12; \tag{3}$$

adding Eq. (2) to Eq. (3) we obtain

$$2x^2=32,$$
$$x^2=16,$$
$$x=\pm 4;$$

subtracting Eq. (3) from Eq. (2), we get

$$2y^2=8,$$
$$y^2=4,$$
$$y=\pm 2.$$

Ans. $\left\{\begin{array}{l} x=\pm 4, \\ y=\pm 2. \end{array}\right.$

This example will afford the student an illustration of the usefulness of particular devices for solving equations, even of a high degree.

19. Given $\left\{\begin{array}{l} x^4-y^4=544 \\ x^2+y^2=34 \end{array}\right\}$, to find x and y.

Ans. $\left\{\begin{array}{l} x=\pm 5, \\ y=\pm 3. \end{array}\right.$

20. Given $\left\{\begin{array}{l} x^4-y^4=1040 \\ x^2-y^2=20 \end{array}\right\}$, to find x and y.

Ans. $\left\{\begin{array}{l} x=\pm 6, \\ y=\pm 4. \end{array}\right.$

21. Given $\left\{\begin{array}{l} x^6-y^6=14896 \\ x^3+y^3=152 \end{array}\right\}$, to find x and y.

OPERATION.

$$x^6-y^6=14896 \tag{1}$$

$$x^3+y^3=152. \tag{2}$$

Dividing Eq. (1) by Eq. (2), we have

$$x^3-y^3=98. \tag{3}$$

Adding Eq. (2) to Eq. (3) we get

$$2x^3 = 250,$$

$$x^3 = 125;$$

extracting cube root, $x = 5;$

Subtracting Eq. (3) from (Eq. (2), we have

$$2y^3 = 54,$$

$$y^3 = 27,$$

$$y = 3.$$

The values of x and y are both *positive*, because the *odd* root of any quantity has the same sign as the quantity itself.

22. Given $\left\{\begin{array}{l} x^6 - y^6 = 4032 \\ x^3 + y^3 = 72 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x = 4, \\ y = 2. \end{array}\right.$

23. Given $\left\{\begin{array}{l} x^6 - y^6 = 728 \\ x^3 - y^6 = 26 \end{array}\right\}$, to find x and y. *Ans.* $\left\{\begin{array}{l} x = 3, \\ y = 1. \end{array}\right.$

PROBLEMS.

1. Find two numbers whose difference is 4 and their product 96. *Ans.* 8 and 12.

2. There are two numbers whose sum is 12 and the sum of their squares is 74; what are the numbers? *Ans.* 7 and 5.

3. There are two numbers whose sum is 13 and the difference of their squares is 91; what are the numbers? *Ans.* 10 and 3.

4. The product of two numbers is 192 and their sum is 28; what are the numbers? *Ans.* 12 and 16.

5. There are two numbers whose sum is 20, and their difference multiplied by their sum gives a product of 80; what are the numbers? *Ans.* 8 and 12.

6. Divide 36 into two such parts that their product shall be 320. *Ans.* 16 and 20.

7. What two numbers are those whose difference is 4, and half their product increased by 6 is equal to the square of the smaller number? *Ans.* 6 and 10.

8. The difference of two numbers is 3, and the sum of their squares is 225; what are the numbers?

Ans. 9 and 12.

9. The sum of the squares of two numbers is 1377, and their quotient is 4; what are the numbers? *Ans.* 9 and 36.

10. The sum of two numbers is 20, and the difference of their squares is 80; what are the numbers? *Ans.* 8 and 12.

11. The sum of the squares of two numbers is 100, and the difference of their fourth powers is 2800; what are the numbers? *Ans.* ± 8 and ± 6.

12. Find two numbers whose sum shall be 61, and the sum of their square roots shall be 11.

OPERATION.

Let $x =$ one of the numbers,

$y =$ the other.

By the conditions, we have $x+y=61,$ (1)

$\sqrt{x}+\sqrt{y}=11;$ (2)

transposing $\sqrt{y}$, we have $\sqrt{x}=11-\sqrt{y};$ (3)

squaring both sides, $x=121-22\sqrt{y}+y;$ (4)

substituting the value of x, as found in Eq. (1),

$121-22\sqrt{y}+y+y=61;$ (5)

transposing and uniting, $2y-22\sqrt{y}=-60,$ (6)

$y-11\sqrt{y}=-30;$ (7)

completing the square, $y-11\sqrt{y}+\frac{121}{4}=\frac{121}{4}-\frac{120}{4}=\frac{1}{4};$ (8)

extracting square root, $\sqrt{y}-\frac{11}{2}=\pm\frac{1}{2},$ (9)

$\sqrt{y}=\frac{11}{2}\pm\frac{1}{2}=6, \text{ or } 5;$ (10)

squaring both sides, $y=36, \text{ or } 25;$ (11)

substituting this value of y in Eq. (3), we have

$\sqrt{x}=11-\sqrt{y}=11-\sqrt{36, \text{ or } 25},$

or $\sqrt{x}=11-(6 \text{ or } 5)=5, \text{ or } 6;$

squaring both sides, $x=25, \text{ or } 36.$

Proof. $x+y=25+36=61,$

$\sqrt{x}+\sqrt{y}=5+6=11.$

13. Find two numbers whose sum shall be 13, and the sum of their square roots shall be 5. *Ans.* 4 and 9.

14. Find two numbers whose difference shall be 28, and the sum of their square roots shall be 14.

Ans. 36 and 64.

SECTION XVI.

RATIO AND PROPORTION.

RATIO.

228. When we compare two quantities of the same kind, we frequently say that one is *twice*, or *three* times, or *four* times, or some number of times, as great as the other. We simply mean that if one of the quantitities were divided by the other, the latter would be contained in the former twice, or three times, or four times, or some number of times.

Thus we say, "15 is three times as great as 5," because 5 is contained 3 times in 15; "12 is four times as great as 3," because 3 is contained 4 times in 12.

This is expressed concisely by saying that the RATIO of 15 to 5 is 3; of 12 to 3 is 4, etc. Hence,

229. RATIO *is the relation, with respect to magnitude. of one quantity to another.*

The RATIO of two quantities is usually indicated by two dots, one placed above the other, between the two quantities; thus, $8:4$ expresses the ratio of 8 to 4; $a:b$ indicates the ratio of a to b.

230. The two quantities compared are the *terms* of the ratio, the first being called the *antecedent*,* the second the *consequent*;† and the two terms taken together constitute a *couplet*.

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* *Antecedent*, from the Latin *antecedo*, signifying *to go before.*

† *Consequent*, from the Latin *consequor*, *to follow after.*

231. RATIO is often expressed in the form of a fraction by making the antecedent the *numerator*, and the consequent the *denominator*; thus,

$$12:4=\frac{12}{4},$$

$$a:b=\frac{a}{b},$$

$$x:y=\frac{x}{y}.$$

EXAMPLES.

1. What is the ratio of 7 to 10? *Ans.* $\frac{7}{10}$.
2. What is the ratio of m to n? *Ans.* $\frac{m}{n}$.
3. What is the ratio of 13 to 19? *Ans.* $\frac{13}{19}$.
4. What is the ratio of $3p$ to $5q$? *Ans.* $\frac{3p}{5q}$.
5. What is the ratio of 18 to 3? *Ans.* ——.
6. What is the ratio of 7 to 12? *Ans.* ——.
7. What is the ratio of (a^2-b^2) to $(a+b)$? *Ans.* $(a-b)$.
8. What is the ratio of 4 to 12? *Ans.* $\frac{1}{3}$.
9. What is the ratio of mx to m? *Ans.* ——.
10. What is the ratio of 21 to 7? *Ans.* ——.
11. What is the ratio of $6x$ to $10x$? *Ans.* $\frac{3}{5}$.
12. What is the ratio of $5b$ to $7b$? *Ans.* ——.
13. What is the ratio of $12ax$ to $4ax$? *Ans.* ——.
14. What is the ratio of (x^2-y^2) to $(x-y)$? *Ans.* ——.

232. In order to find the ratio of two quantities, they must be of the same kind or denomination. Thus, if we were required to find the ratio of 7 shillings to 2 pounds, we could not get it directly, but by reducing the 2 pounds to 40 shillings the ratio will be found to be that of 7 shillings to 40 shillings, or $\frac{7}{40}$.

15. What is the ratio of 4*s.* 2*d.* to 9*d.*? *Ans.* $\frac{50}{9}$.

16. What is the ratio of 2 ft. 3 in. to 10 in.? *Ans.* $\frac{27}{10}$.

17. What is the ratio of 2 hr. to 3 hr. 15 min.? *Ans.* $\frac{8}{13}$.

18. What is the ratio of $\frac{6}{7}$ to $\frac{2}{7}$?

Analysis. It is obvious that 6 *sevenths* is 3 times as much as 2 *sevenths;* hence, the ratio is 3. *Ans.* 3.

19. What is the ratio of $\frac{12}{15}$ to $\frac{3}{15}$?

Analysis. It is plain that 12 *fifteenths* is 4 times as much as 3 *fifteenths;* hence, the ratio is 4. *Ans.* 4.

From the foregoing Analyses we learn the truth that

The ratio of two fractions which have a common denominator is the same as the ratio of their NUMERATORS.

20. What is the ratio of $\frac{18}{19}$ to $\frac{3}{19}$? *Ans.* 6.

21. What is the ratio of $\frac{a}{x}$ to $\frac{b}{x}$? *Ans.* $\frac{a}{b}$.

22. What is the ratio of $\frac{5}{11}$ to $\frac{9}{11}$? *Ans.* ——

23. What is the ratio of $\frac{m}{q}$ to $\frac{n}{q}$? *Ans.* ——.

24. What is the ratio of $\frac{2}{3}$ to $\frac{3}{4}$?

Analysis. In this example we cannot get the ratio immediately, but by reducing the fractions to a *common denominator,* we have $\frac{8}{12}$ and $\frac{9}{12}$, and from what we have seen above the ratio is the same as that of 8 to 9; that is $\frac{8}{9}$. *Ans.* $\frac{8}{9}$.

Hence, to find the ratio between any two fractions,

Reduce them to a common denominator, and take the ratio of their numerators.

25. What is the ratio of $\frac{3}{5}$ to $\frac{2}{3}$? *Ans.* $\frac{9}{10}$.

26. What is the ratio of $\frac{a}{b}$ to $\frac{m}{n}$? *Ans.* $\frac{an}{bm}$.

27. What is the ratio of $\frac{3}{7}$ to $\frac{3}{4}$? *Ans.* $\frac{4}{7}$.

28. What is the ratio of $\frac{5}{9}$ to $3\frac{1}{3}$? *Ans.* $\frac{1}{6}$.

29. Which is the greater, the ratio of 4 to 5 or of 5 to 6?

Analysis. The ratio of 4 to $5 = \frac{4}{5}$; the ratio of 5 to $6 = \frac{5}{6}$. Reducing these to a common denominator, we have $\frac{4}{5} = \frac{24}{30}$, and $\frac{5}{6} = \frac{25}{30}$; and as $\frac{25}{30}$ is greater than $\frac{24}{30}$, we see that the ratio of 5 to 6 is the greater ratio of the two. *Ans.* 5 to 6.

30. Which is the greater ratio, that of 5 to 7 or of 7 to 10? *Ans.* ——.

31 Which is the greater ratio, that of 6 to 8 or of 9 to 11? *Ans.* ——.

32. Which is the greater ratio, that of 3 to 4 or of 9 to 12? *Ans.* ——.

233. Direct ratio is the quotient of the antecedent divided by the consequent. The direct ratio of 15 to 5 is 3; of a to b is $\frac{a}{b}$.

Inverse ratio is the quotient of the consequent divided by the antecedent. The inverse ratio of 4 to 5 is $\frac{5}{4}$; of a to b is $\frac{b}{a}$.

33. What is the direct ratio of 9 to 11? *Ans.* $\frac{9}{11}$.

34. What is the inverse ratio of 7 to 12? *Ans.* $\frac{12}{7}$.

35. What is the inverse ratio of m to n? *Ans.* ——.

36. What is the inverse ratio of $3a$ to $5b$. *Ans.* ——.

37. What is the inverse ratio of $\frac{3}{4}$ to $\frac{4}{5}$? *Ans.* $\frac{16}{15}$.

38. What is the inverse ratio of $\frac{7}{8}$ to $\frac{2}{3}$? *Ans.* $\frac{16}{21}$.

39. What is the inverse ratio of $3\frac{1}{3}$ to $2\frac{2}{5}$? *Ans.* $\frac{18}{25}$.

234. Since a ratio is expressed by a fraction of which the antecedent is the numerator and the consequent the denominator, it follows that the terms of a ratio may be *multiplied* or *divided* just as the terms of a fraction. Hence, the following consequences naturally result—viz.:

I. If the antecedent of a couplet be multiplied by any integral quantity, the ratio will be *increased.*

II. If the consequent be multiplied, the ratio will be diminished.

III. If the antecedent be divided, the ratio will be diminished.

IV. If the consequent be divided, the ratio will be increased.

V. If both the antecedent and the consequent be multiplied, or both be divided, the ratio will not be altered.

It also follows that multiplying the antecedent has the same effect as dividing the consequent, and dividing the antecedent has the same effect as multiplying the consequent.

235. Compound ratio is the ratio of the products of the corresponding terms of two or more simple ratios; thus,

The ratio of $6:2=3$
And the ratio of $8:4=2$
The ratio compounded of these is $48:8=6$

It will be seen that the compound ratio is obtained by multiplying together the two antecedents, and in like manner multiplying together the two consequents, and then finding the ratio of these products.

Again: The ratio of $m:n = \frac{m}{n}$
And the ratio of $a:b = \frac{a}{b}$
Then the compound ratio is $am:bn = \frac{am}{bn}$

40. What is the ratio compounded of 9 : 3 and 15 : 5?
Ans. 9.

41. What is the compound ratio of 8 : 6, of 4 : 3 and 2 : 4?
Ans. $\frac{8}{9}$.

42. What is the compound ratio of $6:4$, of $4:3$ and $7:10$?

Ans. $\frac{7}{5}$.

43. What is the compound ratio of $a:x$, of $x:b$ and $b:y$?

Ans. $\frac{a}{y}$

236. Duplicate ratio is the *square* of the simple ratio.

Thus, the simple ratio of 3 to 4 is $3:4=\frac{3}{4}$;

the duplicate ratio of 3 to 4 is $9:16=\frac{9}{16}$;

the simple ratio of m to n is $m:n=\frac{m}{n}$;

the duplicate ratio of m to n is $m^2:n^2=\frac{m^2}{n^2}$.

237. Triplicate ratio is the *cube* of the simple ratio.

Thus, the simple ratio of 4 to 6 is $4:6=\frac{4}{6}$;

the triplicate ratio of 4 to 6 is $64:125=\frac{64}{125}$;

the simple ratio of a to m is $a:m=\frac{a}{m}$;

the triplicate ratio of a to m is $a^3:m^3=\frac{a^3}{m^3}$.

PROPORTION.

238. When four quantities are so related to each other that the first divided by the second gives the same quotient as the third divided by the fourth, they are said to form a proportion. The numbers 16, 8, 12 and 6 are in proportion, for the first is *twice* as great as the second, and the third is *twice* as great as the fourth.

In the same manner, the numbers 3, 9, 2 and 6 are in proportion, for the first is *one-third* of the second, and the third is *one-third* of the fourth.

239. When four quantities are in proportion, the fact is expressed concisely by saying, *The first is to the second as* *the third is to the fourth.* Thus we say, 16 is to 8 as 12 is to 6, and we write it in this manner:

$$16 : 8 :: 12 : 6.$$

By inspecting this proportion we see that the ratio of 16 to 8 is 2, and the ratio of 12 to 6 is 2; or, in other words, the ratio of the first and second terms is equal to the ratio of the third and fourth terms; *i. e.* $\frac{16}{8} = \frac{12}{6}$. Hence,

240. PROPORTION *is an equality of* RATIOS.

12 : 4 :: 9 : 3 is a PROPORTION, for the ratio of 12 to 4 is 3, and the ratio of 9 to 3 is 3 also.

From the foregoing considerations it is obvious that a proportion is an EQUATION, the ratio of two quantities *uniformly* constituting one member of the *equation*, and the ratio of two other quantities constituting the other member. Hence, from the general principles of *equations*, exactly the same changes that are made on one ratio must be made on the other.

241. A proportion may be written in three different forms, yet all meaning exactly the same thing. Thus, the proportion above may be written,

$$12 : 4 :: 9 : 3,$$

or, $$12 : 4 = 9 : 3,$$

or, $$\frac{12}{4} = \frac{9}{3};$$

and all of them may be read, "The ratio of 12 to 4 is equal to the ratio of 9 to 3;" and any one form may be exchanged for the others.

Again: $$a : b :: c : d,$$

or, $$a : b = c : d,$$

or, $$\frac{a}{b} = \frac{c}{d},$$

mean exactly the same thing, and may be read, "*a* is to *b* as *c* is to *d*," or "The ratio of *a* to *b* is equal to the ratio of *c* to *d*."

242. The first term of a proportion is called the *first antecedent;* the second term, the *first consequent;* the third term, the *second antecedent;* and the fourth term, the *second consequent.* The first and third terms, therefore, are *antecedents,* and the second and fourth are *consequents.*

243. The first and last terms of a proportion are called the *extremes,* and the second and third terms the *means.* Thus, in the proportion $a : b :: c : d$, a and d are the *extremes,* and b and c are the *means;* a and b together constitute the *first couplet,* and c and d the *second couplet.*

244. In the proportion 18 : 6 : : 9 : 3, which are the antecedents? which are the consequents? which are the extremes? which are the means? What is the ratio of the first and second terms? what is the ratio of the third and fourth terms? what is the ratio of the two antecedents? what is the ratio of the two consequents?

245. Since in any proportion the ratio of the first term to the second is always equal to the ratio of the third term to the fourth, we have in the proportion

$$a : b :: c : d,$$

$$\frac{a}{b} = \frac{c}{d};$$

clearing of fractions, $ad = bc.$

But a and d are *extremes,* and b and c are *means;* hence, we reach this very important truth—viz.:

In any proportion the product of the extremes is always equal to the product of the means.

This principle lies at the very foundation of proportion, and is fertile in important results. One of its consequences is, that it furnishes an infallible test of the correctness of a proportion.

To determine whether a proportion is correct, we simply compare the product of the extremes with the product of the means; if the two products are equal, the proportion is right; if unequal, the proportion is wrong.

Ex. 1. Is $3 : 9 :: 6 : 18$ a correct proportion? Why?

Ans. ——.

2. Is $6 : 10 :: 12 : 21$ a correct proportion? Why?

3. Is $(a+b) : (a^2-b^2) :: 1 : (a-b)$ a correct proportion? Why?

4. Is $\frac{3}{4} : \frac{8}{10} :: 60 : 64$ a correct proportion? Why?

5. Is $\frac{4}{5} : \frac{7}{8} :: 96 : 105$ a correct proportion? Why?

6. Is $\frac{2}{3} : \frac{5}{6} :: 185 : 225$ a correct proportion? Why?

7. Is $3x+5 : 2x-3 :: 6x+9 : 4x-5$ a correct proportion? Why?

246. Recurring again to our equation above (viz. $ad = bc$),

I. Dividing both sides by d, we have $a = \dfrac{bc}{d}$.

II. Dividing both sides by c, we have $b = \dfrac{ad}{c}$.

III. Dividing both sides by b, we have $c = \dfrac{ad}{b}$.

IV. Dividing both sides by a, we have $d = \dfrac{bc}{a}$.

From the above we learn this truth—viz.:

If any three terms of a proportion are given, the remaining term can be readily found.

The first of the *four* preceding algebraic expressions may be translated thus:

In any proportion the first term is equal to the product of the second and the third terms, divided by the fourth.

Or we might translate the expression in this way:

In any proportion the first extreme is equal to the product of the two means, divided by the last extreme.

In a similar manner let the student translate the three remaining formulas.

In the fourth formula the pupil will readily recognize the principle underlying operations in the "Rule of Three" in Arithmetic, where we have *three* terms given to find a *fourth*. The rule as practised in Arithmetic is to "multiply the second and third terms together, and divide the product by the first;" which is equivalent to saying that

The last extreme is equal to the product of the two means, divided by the first extreme.

Ex. 1. Given the first three terms of a proportion, 32, 24 and 20, to find the fourth term.

Let $x =$ the fourth term;
then, $32 : 24 :: 20 : x$,
$32x = 480$, (245)
$x = \frac{480}{32} = 15$. *Ans.* 15.

2. Given the first, the second and the fourth terms of a proportion, 16, 48 and 54, to find the third term.

Let $x =$ the third term;
then, $16 : 48 :: x : 54$,
$48x = 864$, (245)
$x = \frac{864}{48} = 18$. *Ans.* 18.

3. Given the first, the third and the fourth terms of a proportion, 162, 256 and 384, to find the second term.
Ans. 243.

4. Given the second, the third and the fourth terms of a proportion, 154, 216 and 252, to find the first term.
Ans. 132.

5. Given the first, the second and the fourth terms of a proportion, 72, 252 and 525, to find the third term.
Ans. 150.

6. Given the first three terms of a proportion, $6xy$, $9ab$ and $18x^2y$, to find the fourth term. *Ans.* $27abx$.

7. Given the first, the second and the fourth terms of a proportion, $21x^2$, $35y$ and $85xy$, to find the third term.
Ans. $51x^3$.

247. While *four* terms are necessary to form a proportion, yet a proportion may be formed by *three quantities*. In such cases one of the quantities is *repeated*, so as to be a consequent of the first couplet and the antecedent of the second couplet: thus,

$$4 : 6 :: 6 : 9,$$

or $$a : x :: x : b.$$

Here, 6 in the first proportion and x in the second are consequents of the first couplet and antecedents of the second. Quantities thus situated are called MEAN PROPORTIONALS.

In the proportion $a : x :: x : b$,

if we multiply extremes and means, we have

$$x^2 = ab,$$

and $$x = \sqrt{ab}.$$

Hence, *To find a mean proportional between two quantities extract the square root of their product.*

Ex. 1. Find a mean proportional between 12 and 3.

Let $x =$ the mean;

then, $12 : x :: x : 3,$

$x^2 = 36,$

$x = \sqrt{36} = 6.$

Our proportion will then read, $12 : 6 :: 6 : 3$.

2. Find a mean proportional between 16 and 4. *Ans.* 8.

3. Find a mean proportional between 6 and 24.
Ans. 12.

4. What is a mean proportional between 16 and 25?
Ans. 20.

5. What is a mean proportional between 36 and 64?
Ans. 48.

6. Find a mean proportional between 36 and 144.

Ans. 72.

7. What is a mean proportional between $\frac{3}{4}$ and 12?

Ans. 3.

8. Find a mean proportional between $\frac{2}{3}$ and 54. *Ans.* 6.

9. Find a mean proportional between $\frac{2}{5}$ and $\frac{8}{20}$. *Ans.* $\frac{4}{10}$.

248. Let us take the equation $ad = bc$.

Dividing both sides by b, we have

$$\frac{ad}{b} = c;$$

dividing this last expression by d, we have

$$\frac{a}{b} = \frac{c}{d},$$

or, $a : b :: c : d$.

From which we learn the important truth—viz.:

If the product of any two quantities is equal to the product of two other quantities, two of them may be made the extremes, and the other two the means, of a proportion.

Ex. 1. Convert $ax = by$ into a proportion.

Ans. $a : b :: y : x$.

2. Change $bm = cn$ into a proportion. *Ans.* $b : c :: n : m$.

3. Convert $mp = nq$ into a proportion. *Ans.* ——.

4. Change $3x = 5y$ into a proportion. *Ans.* ——.

5. Convert $xy = ab$ into a proportion. *Ans.* ——.

6. Change $cd = y^2$ into a proportion. *Ans.* ——.

7. Convert $a^2 - b^2 = x^2 - y^2$ into a proportion.

Ans. $a - b : x - y :: x + y : a + b$.

8. Change $m^2 - n^2 = c^2 - d^2$ into a proportion. *Ans.* ——.

9. Convert $4a^2 - 9b^2 = 25r^2 - 16y^2$ into a proportion.

Ans. ——

10. Change $a^2 - n^2 = b^2$ into a proportion. *Ans.* ——.

11. Convert $ax = y$ into a proportion.
Ans. $a : y :: 1 : x$; or, $a : \sqrt{y} :: \sqrt{y} : x$.

12. Change $bm = x$ into a proportion. *Ans.* ——.

13. Convert $ad = c$ into a proportion. *Ans.* ——.

14. Change $p^2 - q^2 = (a + b)^2$ into a proportion.
Ans. ——.

15. Convert $(ax - ab) = (r^2 + rs)$ into a proportion.
Ans. ——.

16. Change $y^2 + y = a^2 - a$ into a proportion. *Ans.* ——.

249. Since the product of the extremes being equal to the product of the means affords an unfailing proof of the correctness of a proportion, it follows as a necessary consequence that

The terms of a proportion may be written in ANY *order that will leave the product of the extremes equal to the product of the means.*

Let us take the proportion

$a : b :: c : d.$ (I.)
Then, $a : c :: b : d,$ (II.)
or, $b : a :: d : c,$ (III.)
and $b : d :: a : c,$ (IV.)
$c : a :: d : b,$ (V.)
$c : d :: a : b,$ (VI.)
$d : c :: b : a,$ (VII.)
$d : b :: c : a.$ (VIII.)

Here we have thrown the proportion into *eight* different forms, and we know all these forms to be correct, for in each instance, by multiplying extremes and means, we find $ad = bc$.

The same principle can be illustrated *numerically*.

Let us take the proportion

12 : 6 :: 4 : 2. (I.)
Then, 12 : 4 :: 6 : 2, (II.)
or, 6 : 12 :: 2 : 4, (III.)
and 6 : 2 :: 12 : 4, (IV.)
4 : 12 :: 2 : 6, (V.)
4 : 2 :: 12 : 6, (VI.)
2 : 4 :: 6 : 12, (VII.)
2 : 6 :: 4 : 12. (VIII.)

Here also we have *eight* proportions, and we know that each is correct, for in each case we have $12 \times 2 = 6 \times 4$, or $24 = 24$.

250. Hence, we conclude that

In any proportion the order of the means, or of the extremes, or of the terms of both couplets, may be changed without destroying the proportion.

Of the above changes, the one marked II. is called ALTERNATION. We see that it consists simply in changing the order of the means; hence,

251. ALTERNATION* *is changing the order of the means in a proportion.*

If $m : n :: p : q$,
then, by alternation, $m : p :: n : q$.
If 9 : 3 :: 6 : 2,
then, by alternation, 9 : 6 :: 3 : 2.

The change indicated by III. is called INVERSION. It consists in changing the order of the terms of both couplets; hence,

252. INVERSION† *is changing the order of the terms of both couplets.*

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* ***Alternation,*** **from the Latin** ***alterno, to interchange, to change one with another.***

† ***Inversion,*** **from the Latin** ***inverto, to turn about, to turn round.***

It $m : n :: p : q$,

by inversion, $n : m :: q : p$.

If $9 : 3 :: 6 : 2$,

by inversion, $3 : 9 :: 2 : 6$.

253. If $a : b :: c : d$, (I.)

and $m : n :: c : d$, (II.)

then, $a : b :: m : n$.

From Proportion (I.), $\frac{a}{b} = \frac{c}{d}$;

from Proportion (II.), $\frac{m}{n} = \frac{c}{d}$;

then, $\frac{a}{b} = \frac{m}{n}$, (Ax. II.)

or, $a : b :: m : n$.

From which we learn the truth that

If in any two proportions the third and the fourth terms are the same, then a third proportion can be formed out of the first and the second terms of each.

In like manner, if the *first* and the *second* terms in two proportions are the same, then a third proportion can be formed out of the *third* and the *fourth* terms of each.

Again: If $a : b :: c : d$, (I.)

and $a : m :: c : n$, (II.)

then, $b : d :: m : n$.

By alternation, $a : c :: b : d$,

and $a : c :: m : n$;

therefore, by the previous part of this article,

$$b : d :: m : n.$$

From this we learn the truth that

If in any two proportions the ANTECEDENTS *are the same, then a third proportion can be formed out of the* CONSEQUENTS *of each.*

In a similar manner, if the like terms are the *consequents*, then a new proportion can be formed out of the *antecedents*.

Furthermore, if the *antecedents* of the first proportion are the same with the *consequents* of the second, then a new proportion can be formed out of the *consequents* of the first and the *antecedents* of the second.

Ex. 1. Given $\left\{\begin{matrix} m:n::x:y \\ m:n::p:q \end{matrix}\right\}$, to derive a third proportion
Ans. $x:y::p:q$.

2. Given $\left\{\begin{matrix} c:d::p:q \\ m:n::p:q \end{matrix}\right\}$, to deduce a new proportion.
Ans. $c:d::m:n$.

3. Given $\left\{\begin{matrix} a:b::c:d \\ c:d::m:n \end{matrix}\right\}$, to derive a new proportion.
Ans. $a:b::m:n$.

4. Given $\left\{\begin{matrix} f:g::a:b \\ f:m::a:c \end{matrix}\right\}$, to obtain a new proportion.
Ans. $g:b::m:c$.

5. Given $\left\{\begin{matrix} a:b::c:d \\ m:b::n:d \end{matrix}\right\}$, to deduce a new proportion.
Ans. $a:c::m:n$.

6. Given $\left\{\begin{matrix} m:n::p:q \\ a:m::b:p \end{matrix}\right\}$, to derive a new proportion.
Ans. $a:n::b:q$.

NOTE.—The student is expected to show how the above answers are obtained.

254. In the practical applications of proportion we have frequent occasion to produce alterations of the terms by *multiplication* or *division*. In all such changes we must be careful that the *equality between the ratios* shall always be preserved.

If $\qquad a:b::c:d,$

then, $\qquad ad=bc. \quad (245)$

Multiplying both sides by m, which does not destroy the equality, we have

$$mad=mbc.$$

Converting this equation into a proportion (248), we have

$$ma : mb :: c : d, \qquad \text{(I.)}$$

or $$a : b :: mc : md, \qquad \text{(II.)}$$

or $$ma : b :: mc : d, \qquad \text{(III.)}$$

or $$a : bm :: c : dm. \qquad \text{(IV.)}$$

From Proportions (I.) and (II.) we learn that both terms of a couplet may be *multiplied* by the same quantity without destroying the proportion; and from (III.) and (IV.) we see that the two antecedents or the two consequents may be multiplied without destroying the proportion.

Again: If $a : b :: c : d$,
then, $ad = bc$.

Dividing both sides by m, we have

$$\frac{ad}{m} = \frac{bc}{m}.$$

Converting this equation into a proportion (248), we have

$$\frac{a}{m} : \frac{b}{m} :: c : d, \qquad \text{(V.)}$$

or $$a : b :: \frac{c}{m} : \frac{d}{m}, \qquad \text{(VI.)}$$

or $$\frac{a}{m} : b :: \frac{c}{m} : d, \qquad \text{(VII.)}$$

or $$a : \frac{b}{m} :: c : \frac{d}{m}. \qquad \text{(VIII.)}$$

From Proportions (V.) and (VI.) we learn that both terms of a couplet may be *divided* by the same quantity without destroying the proportion; and from (VII.) and (VIII.) it is obvious that the two antecedents or the two consequents may be *divided* without destroying the proportion.

QUESTIONS.

What effect is produced on a proportion by multiplying the two antecedents?

Ans. No effect.

How do we know this?

Ans. Because the product of the extremes still remains equal to the product of the means.

What effect is produced on the *ratios* by multiplying the antecedents?

Ans. Each ratio is equally increased.

What effect on the proportion has multiplying the consequents?

What effect has it on the ratios?

What effect on the proportion has dividing the antecedents?

What effect has it on the ratios?

NOTE.—Questions like the above will readily occur to the teacher; their effect will be to quicken the perceptions of the pupil and give him a more thorough grasp of the subject.

255. If $a : b :: c : d,$

then, $\frac{a}{b} = \frac{c}{d}.$

Adding 1 to each of these equals, we have

$$\frac{a}{b}+1 = \frac{c}{d}+1,$$

and $$\frac{a+b}{b} = \frac{c+d}{d},$$

or $a+b : b :: c+d : d.$

Numerically, if $12 : 8 :: 6 : 4,$

then, $12+8 : 8 :: 6+4 : 4,$

or $20 : 8 :: 10 : 4.$

Hence, we learn—

If four quantities are in proportion, the sum of the first and *second will be to the second as the sum of the third and fourth will be to the fourth.*

This operation is called COMPOSITION.*

256. What has just been shown in regard to *adding* the terms of a proportion is equally true in reference to *subtracting* them.

If $$a : b :: c : d,$$

then, $$\frac{a}{b} = \frac{c}{d}.$$

Subtracting 1 from each of these equals, we have

$$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

and $$\frac{a-b}{b} = \frac{c-d}{d},$$

or, $$a-b : b :: c-d : d.$$

Numerically, if $$12 : 8 :: 6 : 4,$$

then, $$12-8 : 8 :: 6-4 : 4,$$

or $$4 : 8 :: 2 : 4.$$

This operation is usually called DIVISION.†

257. If $$a : b :: c : d,$$

and $$m : n :: p : q,$$

then, $$am : bn :: cp : dq;$$

for $$\frac{a}{b} = \frac{c}{d},$$

and $$\frac{m}{n} = \frac{p}{q}.$$

Multiplying these equals together, we have

$$\frac{am}{bn} = \frac{cp}{dq}.$$

---

* *Composition,* from the Latin *compono,* meaning *to put together.*

† Although sustained by the usage of the great majority of authors, we cannot regard this term as happily chosen to express the operation intended.

Hence, [illegible]

Numerically, if [illegible]

and [illegible]

then, [illegible]

or [illegible]

Hence, we learn—

If two or more proportions be multiplied together, term by term, the resulting products will be proportional.

This operation is called compounding proportions.

That this principle is correct is evident from the fact that the first ratio in each proportion is equal to the second ratio in the same proportion; and when we multiply the corresponding terms together, we are really *multiplying equals by equals*; and hence, if the terms were proportional at first, the products must be proportional.

The reverse of this principle may readily be shown to be true—viz.:

If the terms in one proportion be divided by the corresponding terms in another, the quotients will be proportional.

258. If $a : b : : c : d$,

then, $\frac{a}{b} = \frac{c}{d}$.

Raising each of these quantities to the second power, we have

$$\frac{a^2}{b^2} = \frac{c^2}{d^2},$$

or $a^2 : b^2 : : c^2 : d^2$.

In the same manner we may prove it of *any* power.

Numerically, if $6 : 3 : : 4 : 2$,

then, $6^2 : 3^2 : : 4^2 : 2^2$,

or $36 : 9 : : 16 : 4$.

Again: Since $a : b : : c\ d$,

then, $\frac{a}{b} = \frac{c}{d}$.

Extracting the square root of each of these equals, we have

$$\frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{c}}{\sqrt{d}},$$

or $$\sqrt{a}:\sqrt{b}::\sqrt{c}:\sqrt{d}.$$

In a similar manner we may prove it of *any* root.

Numerically, if $64:16::36:9$,

then, $\sqrt{64}:\sqrt{16}::\sqrt{36}:\sqrt{9}$,

or $8:4::6:3$.

From the foregoing illustrations we learn that

If four quantities are proportional, their like powers or like roots will be proportional.

259. Let a, b, c, d, m, n, p and q be any number of proportional quantities, such that

$a:b::c:d::m:n::p:q$; or, which is the same thing,

let $a:b::c:d$, (I.)

and $a:b::m:n$, (II.)

$a:b::p:q$. (III.)

Then will $a:b::a+c+m+p:b+d+n+q$.

For from Proportion (I.), $ad=bc$,

and from (II.), $an=bm$,

from (III.), $aq=bp$.

Adding these equals, we have

$$ad+an+aq=bc+bm+bp.$$

Now, adding ab to both sides of this last equation, we have

$$ab+ad+an+aq=ab+bc+bm+bp,$$

or $$a(b+d+n+q)=b(a+c+m+p).$$

Converting this equation into a proportion (248),

$$a:b::a+c+m+p:b+d+n+q.$$

Numerically, if $12:6::10:5::8:4::6:3$,

then, $12:6::12+10+8+6:6+5+4+3$,

or $12:6::36:18$.

Hence, it follows—

If there is any number of proportional quantities, all having *the same ratio, then any one antecedent will be to its consequent as the sum of all the antecedents to the sum of all the consequents.*

PROBLEMS IN PROPORTION.

1. Find a mean proportional between 72 and 18. *Ans.* 36.

2. What is a mean proportional between 81 and 64? *Ans.* 72.

3. The first, third and fourth terms of a proportion are 256, 648 and 81; what is the second term? *Ans.* 32.

4. The last three terms of a proportion are 1015, 52 and 364; what is the first term? *Ans.* 145.

5. What two numbers are those whose product is 240, and the greater is to the less as 5 to 3? *Ans.* 12 and 20.

6. Divide 36 into two such parts that the squares of the parts shall be to each other as 25 to 16. *Ans.* 20 and 16.

7. There are two numbers which are to each other as 3 to 4, and if 8 be added to each the sums will be to each other as 4 to 5; what are the numbers? *Ans.* 24 and 32.

8. Two men together gained 150 dollars by trading, and the share of the first was to that of the second as the share of the second was to 40; what was the share of each? *Ans.* $90 and $60.

9. What number is that to which if 3, 9 and 17 be severally added the first sum will be to the second as the second to the third? *Ans.* 15.

10. Divide $2040 among three persons, A, B and C, so that A's share shall be to B's as 3 to 4, and B's to C's as 2 to 5. *Ans.* A's, $360; B's, $480; C's, $1200.

11. Find two numbers the greater of which shall be to the less as 3 to 2, and their sum shall be to their product as 5 to 36. *Ans.* 18 and 12.

12. A farmer has two square barley-lots, a side of one of which is 6 rods longer than the other, and their areas are to each other as 9 to 4; what are the lengths of the sides?

Ans. 18 and 12 rods.

13. Divide 40 into two such parts that the squares of those parts may be to each other as 36 to 16.

Ans. 24 and 16.

14. What two numbers are those whose difference is to their sum as 3 to 5, and their sum is to their product as 5 to 16? *Ans.* 16 and 4.

15. Divide 30 into two such parts that their product shall be to the sum of their squares as 6 to 13. *Ans.* 12 and 18.

16. A and B gained $975 by trading, which they divided in such a manner that A's share was to B's share as $\frac{1}{2}$ to $\frac{1}{3}$; what was the share of each? *Ans.* A, $585; B, $390.

17. There are two numbers which are to each other as 4 to 9, and 30 is a mean proportional between them; what are the numbers? *Ans.* 20 and 45.

18. A gentleman dying left to his wife, his daughter and his son $8712, to be distributed in such a manner that the share of the wife was to be to that of the daughter as 4 to 3, and the share of the daughter was to be to that of the son as 3 to 2; what were their respective shares?

Ans. Wife, $3872; daughter, $2904; son, $1936.

SECTION XVII.

ARITHMETICAL PROGRESSION.

260. AN ARITHMETICAL PROGRESSION is a series of quantities which *increase* or *decrease* by a *common difference*

When they increase, they form an ASCENDING SERIES.

When they decrease, they form a DESCENDING SERIES.

Thus, 3, 5, 7, 9, 11, etc. is an ASCENDING series.
12, 10, 8, 6, 4, etc. is a DESCENDING series.
$a, a+d, a+2d, a+3d$, etc. is an *ascending* series.
$a, a-d, a-2d, a-3d$, etc. is a *descending* series.

261. The quantities which form the series are called TERMS. The first and the last terms are called EXTREMES, and the remaining terms the MEANS.

262. In arithmetical progressions five quantities enter into calculation:

I. The first term;
II. The last term;
III. The common difference;
IV. The number of terms;
V. The sum of the series.

These quantities are so related that if any *three* of them are known the other *two* can be found.

263. In the solution of questions in arithmetical progression several different cases may occur, the most important of which we propose to consider.

CASE I.

264. The First Term, the Common Difference and the Number of Terms being given, to find the LAST TERM.

Let us take the *ascending* series, of which a is the first term and d is the common difference; our series will be—

$$a, (a+d), (a+2d), (a+3d), (a+4d), (a+5d), \text{etc.}$$

In this series we see that the *second* term is equal to the first term + the common difference; the *third* term equals the first + *twice* the common difference; the *fourth* equals the first + *three* times the common difference; and, in general, *any* term equals the first term + the common difference multiplied by the number of terms, less 1.

If $l =$ the last term, and n represent the number of terms, we shall have, universally,

$$l = a + (n-1)d; \text{ that is,}$$

265. *In any arithmetical progression the last term is equal to the first term + the common difference multiplied by the number of terms, less 1.*

When the series is *descending*, the product of the common difference by the number of terms must be *subtracted* from the first term.

NOTE.—It must be remembered that l may represent any term whatever in the series; for at whatever term we stop, that term, for the time, may be regarded as the *last*.

EXAMPLES.

1. The first term of an ascending series is 3, the common difference 4, and the number of terms 10; what is the last term?

$$l = a + (n-1)d.$$

Here $a = 3$, $d = 4$, and $n = 10$;

hence, $l = 3 + (10-1)4 = 3 + 9 \times 4 = 3 + 36 = 39$. *Ans.* 39.

2. What is the twelfth term in a series whose first term is 5 and common difference 3? *Ans.* 38.

3. What is the fifteenth term in a series whose first term is 7 and common difference 4? *Ans.* 63.

4. What is the twentieth term in a series whose first term is 2 and common difference 5? *Ans.* 97.

5. What is the twenty-fourth term in a series whose first term is 6 and common difference 3? *Ans.* 75.

6. What is the thirtieth term in a series whose first term is 4 and common difference 6? *Ans.* 178.

7. A grocer bought 10 melons, paying 15 cents for the first, 20 cents for the second, 25 cents for the third, and so on; what did he pay for the last? *Ans.* 60 cents.

8. The first term of a descending series is 375 and the common difference 9; what is the fortieth term? *Ans.* 24.

9. A man travelled 15 days. The first day he went 20 miles, the second 22 miles, the third 24 miles, and so on. How far did he travel on the last day? *Ans.* 48 miles.

10. What is the 225th term of a descending progression whose first term is 300 and common difference $\frac{1}{2}$?

Ans. 188.

11. A laborer makes 20 weekly deposits in a savings bank. The first week he deposits $\frac{3}{4}$ of a dollar, the second week he deposits $1\frac{1}{2}$ dollars, the third week $2\frac{1}{4}$ dollars; what sum does he deposit the last week? *Ans.* 15 dollars.

CASE II.

266. The First Term, the Last Term and Number of Terms being given, to find the COMMON DIFFERENCE.

By the formula in 264 we have

$$l = a + (n-1)d;$$

transposing a, we have $l - a = (n-1)d$;

dividing by $(n-1)$, we have

$$\frac{l-a}{n-1} = d = \text{the common difference.}$$

Ex. 1. If the first term of a series is 5, the last term 41, and the number of terms 10, what is the common difference?

$$d = \frac{l-a}{n-1}.$$

Here, $a = 5$, $l = 41$, and $n = 10$;

then, $d = \frac{41-5}{10-1} = \frac{36}{9} = 4$ *Ans.*

2. If the last term is 72, the first term 7, and the number of terms 14, what is the common difference? *Ans.* 5.

3. If the extremes of a series are 7 and 103, and the number of terms is 17, what is the common difference?

Ans. 6.

4. A man travels 10 days on a railroad, going 104 miles the first day and 284 miles the last day, increasing each day's travel by an equal difference; what is the daily increase? *Ans.* 20 miles.

5. A man makes 12 weekly deposits in a savings bank. The first week he deposits $\frac{3}{4}$ of a dollar, and the last week he deposits 9 dollars; what is the weekly difference of deposits? *Ans.* $\frac{3}{4}$ of a dollar.

6. Find 6 arithmetical means between 4 and 39.

Analysis. Here we *seem* to have only *two* terms given (viz. the first and the last), but, in fact, we have the *number* of terms also.

The *whole* number of terms in an arithmetical progression consists of the *two* extremes + the means; and as the means in this example are 6, the *whole* number of terms must be 8. We are therefore required to find only the common difference, by which we can fill out the series; hence,

$$d=\frac{l-a}{n-1}=\frac{39-4}{8-1}=\frac{35}{7}=5, \text{ the common difference.}$$

The series is 4, 9, 14, 19, 24, 29, 34, 39.

7. Find 8 arithmetical means between 11 and 47.
Ans. Com. diff. is 4.

8. Find 10 arithmetical means between 3 and 36.
Ans. C. D. is 3.

9. Find 6 arithmetical means between $\frac{3}{4}$ and 6.
Ans. C. D. is $\frac{3}{4}$.

10. Find 6 arithmetical means between $\frac{1}{3}$ and 5.
Ans. C. D. is $\frac{2}{3}$.

CASE III.

267. The Last Term, the Common Difference and the Number of Terms being given, to find the FIRST TERM.

By the formula in **264**, we have

$$l=a+(n-1)d.$$

By transposing $(n-1)d$ we obtain

$$l-(n-1)d=a=\text{the first term.}$$

Ex. 1. The last term of an ascending series is 29, the common difference 3, and the number of terms 9; what is the first term?

$$a = l - (n-1)d.$$

Here, $l = 29$, $n = 9$, and $d = 3$;

hence, $a = 29 - (9-1)3 = 29 - 8 \times 3 = 29 - 24 = 5$, first term.

2. The last term of a series is 34, the common difference 3, and the number of terms 10; what is the first term? *Ans.* 7.

3. A man has 9 children, whose ages form an arithmetical progression. The age of the oldest is 29 years, and the common difference of their ages is $2\frac{1}{2}$ years; what is the age of the youngest? *Ans.* 9 years.

4. A gentleman travelled 8 days, each day increasing his distance 4 miles, and on the last day he went 45 miles; how far did he travel the first day? *Ans.* 17 miles.

5. A farmer bought 10 cows, paying $3\frac{1}{2}$ dollars more in succession for each one until he paid 50 dollars for the last one; what was the price of the first? *Ans.* $18\frac{1}{2}$ dollars.

6. A merchant has 13 pieces of muslin, the lengths of which are in arithmetical progression, each piece increasing regularly $2\frac{3}{4}$ yards until the longest measures 56 yards; what is the length of the shortest piece? *Ans.* 23 yards.

CASE IV.

268. The Extremes and Common Difference being given, to find the NUMBER of Terms.

By 264, $l = a + (n-1)d$,

transposing (a), $l - a = (n-1)d$;

dividing by d, $\frac{l-a}{d} = n - 1$;

transposing (-1), $\frac{l-a}{d} + 1 = n =$ number of terms.

Ex. 1. The extremes of an arithmetical series are 74 and 8, and the common difference is 6; what is the number of terms?

$$n=\frac{l-a}{d}+1.$$

Here, $l=74$, $a=8$, and $d=6$;

hence, $n=\frac{74-8}{6}+1=\frac{66}{6}+1=11+1=12=$ number of terms.

2. The first term of a series is 32, the last term 87, and the common difference 5; what is the number of terms?
Ans. 12.

3. A basket of plums was divided among a certain number of school-girls, each receiving 10 more than fell to the share of the one who preceded her. The first received 50 and the last 100 plums; how many girls were there?
Ans. 6 girls.

4. A man paid a note by instalments, each time paying 20 dollars more than at the preceding payment. The first payment was 50 and the last 190 dollars; how many payments did he make? *Ans.* 8.

5. In a certain family the ages of the children are in arithmetical progression, the common difference being $2\frac{1}{3}$ years. The oldest is 19 and the youngest 5 years of age; how many children are there? *Ans.* 7.

CASE V.

269. The Extremes and Number of Terms being given, to find the SUM of the Series.

Ex. 1. What is the sum of the series 3, 7, 11, 15, 19 and 23?

Analysis. We might find the sum of the series simply by adding the numbers together, but in an extended progression this process would be exceedingly tedious. In any series the sum of the terms will be the same in whatever order they are written.

Suppose we take the series in the above example, writing under it the same series in an inverted order, then let us add the several terms in the same column.

Let s = sum of each series; we shall then have

$$\begin{aligned} s &= 3+7+11+15+19+23 = 78 = \text{sum of first series,} \\ s &= 23+19+15+11+7+3 = 78 = \text{sum of inverted series.} \\ \hline 2s &= 26+26+26+26+26+26 = 156 = \text{sum of both series.} \end{aligned}$$

We readily perceive that the sum of the terms in the *two* series is equal to 26 multiplied by 6, which is 156. But 26 is the sum of the extremes, and 6 is the number of terms. Since the sum of the extremes, multiplied by the number of terms, gives *twice* the series, it is obvious that if we divide this product by 2 we shall have the sum of the single series = 78.

Or, to present it in a more general form:

Let us take the series

$$a,\ (a+d),\ (a+2d),\ (a+3d),\ (a+4d),$$

and let s = the sum of the series. We shall then have

$$\begin{aligned} s &= a+(a+d)+(a+2d)+(a+3d)+(a+4d) = \text{first series,} \\ s &= (a+4d)+(a+3d)+(a+2d)+(a+d)+a = \text{inverted series.} \\ \hline 2s &= (2a+4d)+(2a+4d)+(2a+4d)+(2a+4d)+(2a+4d) = \text{sum of both series.} \end{aligned}$$

We here see that the sum of the terms in the double series is equal to $(2a+4d)$ repeated 5 times. But $(2a+4d)$ is the *sum* of the *extremes*, and 5 is the *number* of terms. Now, substituting, according to our notation (**264**), a for the first extreme, l for the last and n for the number of terms, we have

$$2s = (a+l)\times n,$$

and $$s = \left(\frac{a+l}{2}\right)\times n;\ i.\ e.,$$

270. *The sum of the terms is equal to half the sum of the extremes multiplied by the number of terms.*

Ex. 1. The extremes of a series are 15 and 85, and the number of terms is 11; what is the sum of the series?

$$s = \left(\frac{a+l}{2}\right)\times n = \frac{15+85}{2}\times 11 = \frac{100}{2}\times 11 = 50\times 11 = 550\ \textit{Ans.}$$

2. The first term of a series is 23, the last term 119,

and the number of terms 17; what is the sum of the series? *Ans.* 1207.

3. The extremes of an arithmetical progression are 13 and 104, and the number of terms is 14; what is the sum of the series? *Ans.* 819.

4. A man made 10 weekly deposits in a bank, each being in arithmetical progression. The first week he deposited $15, and the last deposit was $60; what was the sum deposited? *Ans.* $375.

5. A stone falling freely under the influence of gravity descends $16\frac{1}{12}$ feet in the first second and $305\frac{7}{12}$ feet in the tenth second, increasing $32\frac{1}{6}$ feet in each second; how far will it fall in 10 seconds? *Ans.* $1608\frac{1}{3}$ feet.

6. A clerk agreed to serve a merchant for a year, receiving as his wages one cent the first day, 2 cents the second day, 3 cents the third day, and so on. Counting the year as 313 working days, what was the clerk's salary? *Ans.* $491.41.

271. The two equations,

$$l = a + (n-1)d, \qquad \text{(I.)}$$

and

$$s = \left(\frac{a+l}{2}\right)n, \qquad \text{(II.)}$$

should be carefully remembered by the student, for from these all the formulæ belonging to arithmetical progression are deduced. They embrace *five* quantities, a, l, n, d, s, of which any three being given, the other two can be found.

Twenty different cases may arise for solution, *five* of the more important of which we have already explained.

We do not propose to consider the *fifteen* remaining cases in detail, but simply to give the student a little insight into the method of determining each of them for himself. We will illustrate two more cases.

CASE VI.

272. Given l, n, s, to find a; or, in other words, given the Last Term, the Number of Terms and the Sum of the Terms, to find the FIRST TERM.

Analysis. By reference to Eq. II., **271**, we perceive that the *four* quantities, l, n, s and a, are embraced in that equation; hence, we can find (a) from that equation alone.

$$s=\left(\frac{a+l}{2}\right)n;$$

clearing of fractions, we have

$$2s=(a+l)n;$$

dividing by n, $$\frac{2s}{n}=a+l;$$

transposing l, $$\frac{2s}{n}-l=a=\text{the first term.}$$

Ex. 1. The sum of an arithmetical progression is 144, the last term 32, and the number of terms 8; what is the first term? *Ans.* 4.

2. The sum of a series is 414, the last term 62, and the number of terms 12; what is the first term? *Ans.* 7.

3. The sum of a series is 517, the last term 82, and the number of terms 11; what is the first term? *Ans.* 12.

CASE VII.

273. Given a, n and s, to find d; or, given the First Term, the Number of Terms, and the Sum of the Terms, to find the COMMON DIFFERENCE.

Analysis. By reference to the two fundamental equations (I.) and (II.), **271**, we see that neither of them separately contains the *four* quantities a, n, s and d. Hence it will be necessary to combine the two, so as to deduce the value of the quantity required. By inspection, we see that the quantity l is left out; *i. e.* it is not among those that are *given* or *required* in our example. If we can *eliminate* l, we shall have no difficulty. Now, let us find an expression for the value of l in each of the equations.

In Eq. (I.) we have $l=a+(n-1)d$,

in Eq. (II.) we have $s=\left(\frac{a+l}{2}\right)n$;

clearing of fractions, $2s=(a+l)n$;

dividing by n, $\frac{2s}{n}=a+l$;

transposing a, $\frac{2s}{n}-a=l$.

Putting these two values of l, equal to each other, we have

$$a+(n-1)d=\frac{2s}{n}-a,$$

or, $a+dn-d=\frac{2s}{n}-a$;

clearing of fractions, $an+dn^2-dn=2s-an$;

transposing, $dn^2-dn=2s-2an$,

$$d(n^2-n)=2s-2an,$$

$$d=\frac{2s-2an}{n^2-n}.$$

Ex. 1. The sum of an arithmetical progression is 295, the first term 7, and the number of terms 10; what is the common difference? *Ans.* 5.

The method which we have just explained will give the student an idea how to proceed to find a formula to solve any case that may occur in arithmetical progression. If the *three* quantities *given* and the *one required* are not all found in the *same* equation, it will be necessary to combine the two equations in a manner similar to that we have just shown.

274. Simply with the view of reaching *practical* results, we can dispense with working out these complicated formulas, and solve examples by a few simple artifices.

We will illustrate this point by solving the example last proposed, in which we have given $s=295$, $a=7$ and $n=10$, to find d, the common difference.

Taking the two fundamental equations, (**271**), we have

$$l=a+(n-1)d, \qquad \text{(I.)}$$

and

$$s=\left(\frac{a+l}{2}\right)n; \qquad \text{(II).}$$

clearing (Eq. II.) of fractions, $2s=(a+l)n$.

Now, let us substitute for s, a and n their values as given above, and we obtain $590=(7+l)10$,

or, $590=70+10l$;

transposing, $520=10l$;

then, $l=52$.

Now, substitute this value of l, and also the values of a and n, in Eq. (I.), and we have

$$52=7+(10-1)d,$$

or, $52=7+9d$,

$$45=9d,$$

$$5=d.$$

Here, it will be observed, we get the same numerical value for d as by the regular formula, but the process is much shorter and far more simple.

This little device is capable of being applied to other cases which are ordinarily solved by a complex formula.

As an exercise for the student we append two or three examples:

Ex. 1. The sum of a series is 324, the first term 8, and the number of terms 9; what is the common difference?

Ans. 7.

2. The sum of a series is 252, the first term 4, and the last term 52; what is the common difference? *Ans.* 6.

3. The sum of a series is 270, the common difference 4, and the number of terms 10; what is the first term?

Ans. 9.

4. In an arithmetical progression the sum of the series is 480, the common difference 8, and the number of terms 10; what is the first term? *Ans.* 12.

PROBLEMS IN ARITHMETICAL PROGRESSION.

275. In the examples thus far presented *three* of the quantities have always been given, but problems very frequently occur in which *none* of the quantities are given directly, but are *implied* in the conditions of the question.

In solving problems of this character we represent the quantities by the last letters of the alphabet, but in strict accordance with the principles of arithmetical progression. We will illustrate our meaning more fully by an example:

Ex. 1. The sum of three numbers in arithmetical progression is 21, and the sum of the squares of the two extremes is 106; what are the numbers?

Let $x =$ the first number,

and $y =$ the common difference.

The series will be, $x, (x+y), (x+2y)$.

By the first condition, $x+(x+y)+(x+2y)=21, \quad (1)$

by the second condition, $x^2+(x+2y)^2=106; \quad (2)$

adding terms in Eq. (1), we have $3x+3y=21, \quad (3)$

$$x+y=7. \quad (4)$$

Expanding Eq. (2), we obtain

$$x^2+x^2+4xy+4y^2=106, \quad (5)$$

or,

$$2x^2+4xy+4y^2=106, \quad (6)$$

$$x^2+2xy+2y^2=53. \quad (7)$$

Squaring Eq. (4), $x^2+2xy+y^2=49; \quad (8)$

subtracting (8) from (7), $y^2=4, \quad (9)$

$$y=2. \quad (10)$$

Substituting this value of y in Eq. (4), we have

$$x+2=7,$$

$$x=5.$$

Then, $x=5$ $x+y=7$, $x+2y=9$. *Ans.* 5, 7 and 9.

We migh have solved this problem in a manner slightly different:

Let $x =$ the second of the three numbers,

and $y =$ the common difference.

The series will be $x-y, x, x+y$.

By the first condition, $(x-y)+x+(x+y)=21;$ (1)
by the second condition, $(x-y)^2+(x+y)^2=106;$ (2)
adding terms in Eq. (1), $3x=21,$ (3)
$x=7;$
expanding Eq. (2), $x^2-2xy+y^2+x^2+2xy+y^2=106$
uniting terms, $2x^2+2y^2=106,$
$x^2+y^2=53;$
substituting the value of x, $49+y^2=53,$
$y^2=4,$
$y=2.$

Then, $x-y=5$, $x=7$, $x+y=9$. *Ans.* 5, 7 and 9.

The student cannot fail to notice that the latter method involves a less amount of labor than the other. Whenever, therefore, there are *three* terms, it will be found convenient to represent them thus:

$$(x-y),\ x,\ (x+y).$$

If there are *four* terms, it will frequently very much facilitate the work by representing the series thus·

$$(x-3y),\ (x-y),\ (x+y),\ (x+3y),$$

in which $2y$ represents the common difference.

Should there be *five* terms, it will expedite the solution to represent the series as follows:

$$(x-2y),\ (x-y),\ x,\ (x+y),\ (x+2y).$$

Whenever the *sum* of the terms enters as one of the given conditions, it will be noticed that by employing the notation we have suggested the y's disappear entirely and the value of x is at once obtained.

These little artifices are allowable, inasmuch as they simplify operations and violate no mathematical principle.

2. There are three numbers in arithmetical progression whose sum is 9, and the sum of their cubes is 99; what are the numbers? *Ans.* 2, 3 and 4.

3. Find three numbers in arithmetical progression whose

sum shall be 36, and the sum of the squares of the three numbers shall be 450. *Ans.* 9, 12 and 15.

4. There are three numbers in arithmetical progression whose sum is 21, and the product of the three numbers is 231; what are the numbers? *Ans.* 3, 7 and 11.

5. Find four numbers in arithmetical progression such that the sum of the squares of the two extremes shall be 68, and the sum of the squares of the two means shall be 52. *Ans.* 2, 4, 6 and 8.

NOTE.—Let $(x-3y)$ represent the first, and $2y =$ the common difference.

6. There are four numbers in arithmetical progression whose sum is 28, and the sum of the squares of the second and fourth is 194; what are the numbers?

Ans. 1, 5, 9 and 13.

7. Find four numbers in arithmetical progression whose sum shall be 16, and the sum of the squares of the first and third shall be 26. *Ans.* 1, 3, 5 and 7.

8. There are four numbers in arithmetical progression, and the sum of the first and fourth is 13, and the product of the second and third is 40; what are the numbers?

Ans. 2, 5, 8 and 11.

9. Find four numbers in arithmetical progression such that the sum of the squares of the extremes shall be 272, and the sum of the squares of the means shall be 208.

Ans. 4, 8, 12 and 16.

10. Find five numbers in arithmetical progression such that their sum shall be 30, and the sum of the squares of the two extremes shall be 104. *Ans.* 2, 4, 6, 8 and 10.

NOTE.—Let $x - 2y =$ the first number, and $y =$ the common difference.

11. Find five numbers in arithmetical progression whose

sum shall be 25, and the product of the second and fourth shall be 21. *Ans.* 1, 3, 5, 7 and 9.

12. There are five numbers in arithmetical progression, and their sum is 20, and the sum of the squares of the numbers is 90; what are the numbers? *Ans.* 2, 3, 4, 5 and 6.

SECTION XVIII.

GEOMETRICAL PROGRESSION.

276. ANY series of numbers, like 1, 3, 9, 27, 81, etc., *increasing* by a common *multiplier*, or like 81, 27, 9, 3, 1, *decreasing* by a common *divisor*, is called a GEOMETRICAL PROGRESSION. When the quantities *increase*, they form an ASCENDING series; when they *decrease*, they form a DESCENDING series.

Thus, 5, 15, 45, 135, 405, is an *ascending* series.
405, 135, 45, 15, 5, is a *descending* series.

277. The common multiplier or the common divisor is called the RATIO. The *ratio* may always be regarded as a *multiplier*, either integral or fractional. Thus, in the ascending series above the ratio is 3; in the descending it is $\frac{1}{3}$.

In the series above it will be observed that each succeeding term is found by multiplying the preceding term by the ratio.

278. The quantities forming the series are called TERMS. The first and last terms are called EXTREMES, and the remaining terms the MEANS.

279. If a represent the first term of a geometrical progression, and r the common ratio, the series will be,

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ ar^6,\ \text{etc.}$$

By inspecting this series we see that the *second* term is equal to the first term multiplied by the ratio; the *third* term equals the first multiplied by the *second* power of the ratio; the *fourth* term equals the first multiplied by the *third* power of the ratio; and so *any* term will always be equal to the first term multiplied by that power of the ratio whose index is *one less* than the number of terms.

Ex. 1. What would be the 25th term of the foregoing series? *Ans.* ar^{24}.

2. What would be the 50th term? *Ans.* ——.

3. What would be the 75th term? *Ans.* ——.

4. What would be the 100th term? *Ans.* ——.

5. What would be the mth term? *Ans.* ar^{m-1}.

6. What would be the nth term? *Ans.* ——.

280. In geometrical progression five particulars enter into calculation—viz.:

I. The first term;
II. The last term;
III. The ratio;
IV. The number of terms;
V. The sum of the terms.

These quantities are so related to each other that any *three* of them being given the other *two* can be found.

Twenty different cases may occur. We shall present only those cases which are of most practical utility.

CASE I.

281. The First Term, the Ratio and the Number of Terms being given, to find the LAST TERM.

In **279** it was shown that *any* term of a geometrical progression will always equal the first term multiplied by that power of the ratio whose index is one less than the number of terms. Hence, we have the following

RULE.—*The last term of a geometric series is equal to the first term multiplied by that power of the ratio whose index is one less than the number of terms.*

Let l = the last term, a = the first term, r = the ratio, and n = the number of terms; then,

$$l = ar^{n-1}.$$

EXAMPLES.

1. The first term of a geometric series is 7, the ratio 3, and the number of terms 5; what is the last term?

Here, $a = 7$, $r = 3$ and $n = 5$.

$$l = ar^{n-1} = 7 \times 3^4 = 7 \times 81 = 567.$$

The series is 7, 21, 63, 189, 567. *Ans.* 567.

2. The first term of a series is 972, and the ratio $\frac{1}{3}$; what is the sixth term?

Here, $a = 972$, $r = \frac{1}{3}$ and $n = 6$.

$$l = ar^{n-1} = 972 \times (\tfrac{1}{3})^5 = 972 \times \tfrac{1}{243} = \tfrac{972}{243} = 4.$$

The series is 972, 324, 108, 36, 12, 4. *Ans.* 4.

3. The first term of a series is 5, and the ratio 3; what is the eighth term? *Ans.* 10935.

4. The first term of a geometric series is 1024, the ratio $\frac{1}{2}$, and the number of terms 7; what is the last term? *Ans.* 16.

5. The first term of a series is 3, and the ratio 5; what is the sixth term? *Ans.* 9375.

6. The first term of a series is 9216, the ratio $\frac{1}{4}$, and the number of terms 7; what is the last term? *Ans.* $2\frac{1}{4}$.

7. A man buys 7 books, giving 10 cents for the first, 20 cents for the second, 40 cents for the third, and so on, in geometrical progression; what does he pay for the last book? *Ans.* $6.40.

8. A man travels 8 days, going $2\frac{1}{4}$ miles the first day, $4\frac{1}{2}$ miles the second day, 9 miles the third day, and so on; what is the last day's travel? *Ans.* 288 miles.

9. What is the tenth term of the series 1, 3, 9, 27, etc.? *Ans.* 19683.

10. What is the seventh term of the series 1, 6, 36, 216, etc.? *Ans.* 46656.

CASE II.

282. The Last Term, the Ratio and the Number of Terms being given, to find the FIRST TERM.

Analysis. By **281** we had the formula

$$l = ar^{n-1};$$

dividing by r^{n-1}, we obtain

$$\frac{l}{r^{n-1}} = a = \text{first term.}$$

Ex. 1. The last term of a geometric series is 3072, the ratio 4, and the number of terms 6; what is the first term?

Here, $l = 3072$, $r = 4$ and $n = 6$.

$$a = \frac{l}{r^{n-1}} = \frac{3072}{4^5} = \frac{3072}{1024} = 3 = \text{first term.}$$

The series is 3, 12, 48, 192, 768, 3072.

2. The last term of a series is 9, the ratio $\frac{1}{3}$, and the number of terms 7; what is the first term?

Here, $l = 9$, $r = \frac{1}{3}$ and $n = 7$.

$$a = \frac{l}{r^{n-1}} = \frac{9}{(\frac{1}{3})^6} = \frac{9}{\frac{1}{729}} = 9 \times \frac{729}{1} = 6561 = \text{first term}$$

The series is 6561, 2187, 729, 243, 81, 27, 9.

3. The last term of a series is 15625, the ratio 5, and the number of terms 6; what is the first term? *Ans.* 5.

4. The last term of a series is 6, the ratio $\frac{1}{3}$, and the number of terms 6; what is the first term? *Ans.* 1458.

5. The last term of a series is 15309, the ratio 3, and the number of terms 8; what is the first term? *Ans.* 7.

CASE III.

283. The Two Extremes and the Number of Terms being given, to find the RATIO.

In **281** we had the formula

$$l = ar^{n-1};$$

dividing by a, then $$\frac{l}{a} = r^{n-1};$$

extracting the $(n-1)$th root of both sides, we obtain

$$\left(\frac{l}{a}\right)^{\frac{1}{n-1}} = r = \text{the ratio.}$$

Hence, we derive the following

RULE.—*Divide the last extreme by the first, and extract that root of the quotient denoted by the number of terms less 1.*

EXAMPLES.

1. The extremes of a geometric series are 7 and 567; what is the ratio if there are 5 terms?

Here, $l = 567$, $a = 7$ and $n = 5$.

$$r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \left(\frac{567}{7}\right)^{\frac{1}{4}} = (81)^{\frac{1}{4}} = 3 = \text{ratio.}$$

The series, therefore, is 7, 21, 63, 189, 567.

2. If the first term of a geometric series is 432, the last term 16, and the number of terms 4, what is the ratio?

Here, $a = 432$, $l = 16$ and $n = 4$.

$$r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \left(\frac{16}{432}\right)^{\frac{1}{3}} = \left(\frac{1}{27}\right)^{\frac{1}{3}} = \tfrac{1}{3} = \text{ratio.}$$

The series is 432, 144, 48, 16.

3. The first term of a geometric series is 17, the last term 2125, and the number of terms 4; what is the ratio?

Ans. 5.

4. If the first term of a series is 2304, the last term 9, and the number of terms 5, what is the ratio? *Ans.* $\frac{1}{4}$.

NOTE.—The most expeditious method of extracting the *fourth* root is, first to extract the *square* root, and then extract the square root of the square root already found.

By means of the last formula we can find any number of geometric means between two given numbers. The given numbers may be regarded as the *two extremes* of a geometrical progression, and all that is necessary is to find the *ratio*, as in the preceding examples, and the means can then be supplied by successive multiplication.

5. Find *two* geometric means between 3 and 375.

Analysis. In every progression the *whole* number of terms consists of the *two extremes* and all the *intermediate* terms or *means*. Hence, if there are *two* means the whole number of terms must be *four*. Hence, the formula is

$$r=\left(\frac{l}{a}\right)^{\frac{1}{n-1}}=\left(\frac{375}{3}\right)^{\frac{1}{3}}=(125)^{\frac{1}{3}}=5=\text{ratio.}$$

The ratio is 5, and the series is 3, 15, 75, 375.

6. Find three geometric means between 4 and 324.
Ans. 12, 36 and 108.

7. Find two geometric means between $\frac{1}{4}$ and 16.
Ans. 1 and 4.

8. Find three geometric means between 13 and 1053.
Ans. 39, 117 and 351.

9. Find five geometric means between 11 and 704.
Ans. 22, 44, 88, 176 and 352.

NOTE. The easiest method of extracting the *sixth* root is to *extract the square root*, and then *extract* the *cube root* of the square root thus found.

10. Find three geometric means between $\frac{1}{4}$ and 64.
Ans. 1, 4 and 16.

11. Find five geometric means between 14 and 57344.
Ans. 56, 224, 896, 3584 and 14336.

CASE IV.

284. The First Term, the Ratio, and the Number of Terms being given, to find the SUM of the Terms.

Ex. 1. What is the sum of the series 4, 12, 36, 108, 324 and 972?

Let $s=$ the sum of the series, and we shall have

$$s=4+12+36+108+324+972.$$

Now, let us multiply each side of this equation by the ratio 3, and we have

$$3s=12+36+108+324+972+2916.$$

Subtracting the given series from the latter, we see that all the terms will cancel except the first and the last, and we shall have

$$3s-s=2916-4,$$

or,
$$(3-1)s=2916-4,$$

$$s=\frac{2916-4}{3-1}=\frac{2912}{2}=1456 \quad \textit{Ans.}$$

Again: Let a, ar, ar^2, ar^3, ar^4, etc. represent any geometric series up to n terms; then,

$$s=a+ar+ar^2+ar^3+ar^4 \quad . \quad . \quad . \quad . \quad ar^{n-1};$$

multiplying by r, $rs=ar+ar^2+ar^3+ar^4 \quad . \quad . \quad . \quad . \quad ar^{n-1}+ar^n.$

Subtracting the first equation from the second, we obtain

$$rs-s=ar^n-a,$$

or,
$$(r-1)s=ar^n-a,$$

$$s=\frac{ar^n-a}{r-1}=\frac{lr-a}{r-1}.$$

ar^n is equal to the *last* term in the given series multiplied by the *ratio;* therefore, we can substitute lr for ar^n.

Hence, we have the following

RULE.—*Multiply the last term by the ratio, subtract the first term, and divide the remainder by the ratio less 1.*

EXAMPLES.

2. What is the sum of a geometric series the first term of which is 2, ratio 3, and number of terms 8?

We first find the last term,

$$l=ar^{n-1}=2\times3^7=2\times2187=4374.$$

$$s=\frac{lr-a}{r-1}=\frac{4374\times3-2}{3-1}=\frac{13122-2}{2}=\frac{13120}{2}=6560.$$

3. What is the sum of a geometric series whose first term is 5, last term 1215, and ratio 3? *Ans.* 1820.

4. What is the sum of a geometric series whose first term is 7, last term 7168, and ratio 4? *Ans.* 9555.

5. How large a debt may be discharged in 6 months by paying $6 the first month, $24 the second month, $96 the third month, and so on in geometrical progression?
Ans. $8190.

6. The first term of a geometric series is 8, the ratio 3, and the number of terms 7; what is the sum of the series?
Ans. 8744.

7. A man invested 2 dollars in a lottery, which being lost he invested 6 dollars, and this again being lost he then invested 18 dollars, and this process he repeated 11 times; how large a prize must he draw in order to recover all that he lost? *Ans.* 177146 dollars.

8. Suppose a man should have 5 children, and each of these should have 5 children, and so on in geometrical progression; how many descendants would he have in 8 generations? *Ans.* 488280 descendants.

9. A man purchased a house, agreeing to pay 1 cent for the first door, 4 cents for the second, 16 cents for the third, and so on in geometrical progression; what was the price paid for the house, there being 11 doors?
Ans. $13981.01.

PROBLEMS IN GEOMETRICAL PROGRESSION.

285. Problems in geometrical progression in which none of the quantities are given *directly*, but are implied in the conditions of the question, are solved by representing the unknown quantities in accordance with the principles of progression explained in the preceding pages.

1. Find three numbers in geometrical progression such that their sum shall be 13 and the sum of their squares 91.

Analysis. Let x = the first term and y = the ratio.

The series will be x, xy and xy^2.

By the conditions, $$x+xy+xy^2=13, \tag{1}$$

and $$x^2+x^2y^2+x^2y^4=91; \tag{2}$$

dividing Eq. (2) by Eq. (1), we have

$$x-xy+xy^2=7; \tag{3}$$

subtracting Eq. (3) from Eq. (1), we obtain

$$2xy=6,$$
$$xy=3,$$
$$y=\frac{3}{x}.$$

Substituting this value of y in Eq. (1), we get

$$x+3+\frac{9}{x}=13;$$

clearing of fractions, $$x^2+3x+9=13x;$$

transposing, $$x^2-10x=-9;$$

completing the square,

$$x^2-10x+25=25-9=16;$$

extracting sq. root, $$x-5=\pm 4,$$

$$x=5\pm 4=9, \text{ or } 1,$$

$$y=\frac{3}{x}=\frac{3}{9}, \text{ or } \frac{3}{1}=\frac{1}{3}, \text{ or } 3.$$

If $x=9$, then $y=\frac{1}{3}$; or if $x=1$, then $y=3$.

If $x=1$, then $xy=3$ and $xy^2=9$. *Ans.* 1, 3 and 9.

2. There are three numbers in geometrical progression the sum of which is 28 and the sum of their squares is 336; what are the three numbers? *Ans.* 4, 8 and 16.

3. There are three numbers in geometrical progression such that the sum of the first and last is 30 and the square of the mean is 81; what are the numbers?

Ans. 3, 9 and 27.

4. Find three numbers in geometrical progression such that the sum of the three numbers shall be 62 and the square of the mean shall be 100. *Ans.* 2, 10 and 50.

5. Of four numbers in geometrical progression, the sum of the first and second is 8, and the sum of the third and fourth is 72; what are the numbers? *Ans.* 2, 6, 18 and 54.

6. Find four numbers in geometrical progression such that the sum of the first two shall be 32 and the sum of the last two shall be 288. *Ans.* 8, 24, 72 and 216.

7. A merchant has three pieces of cassimere, the lengths of which are in geometrical progression. The aggregate length of the three pieces is 70 yards, and the longest piece contains 30 yards more than the shortest; what is the length of each piece?

Analysis. Let the series be x, xy, xy^2.

By the conditions, $$x+xy+xy^2=70, \qquad (1)$$

and $$xy^2=x+30, \qquad (2)$$

$$xy^2-x=30, \qquad (3)$$

$$x(y^2-1)=30, \qquad (4)$$

$$x=\frac{30}{y^2-1}. \qquad (5)$$

Resolving Eq. (1) into factors, we have

$$x(1+y+y^2)=70, \qquad (6)$$

$$x=\frac{70}{1+y+y^2}. \qquad (7)$$

Making the two values of x equal to each other, we have

$$\frac{70}{1+y+y^2}=\frac{30}{y^2-1};$$

clearing of fractions, $70y^2-70=30+30y+30y^2$;

transposing, $40y^2-30y=100$;

or, $4y^2-3y=10$;

completing the square, $64y^2-48y+9=160+9=169$;

extracting square root, $8y-3=13$,

$$8y=16,$$

$$y=2.$$

Substituting this value of y in Eq. (5), we obtain

$$x=\frac{30}{y^2-1}=\frac{30}{4-1}=\frac{30}{3}=10.$$

$x=10$, $xy=20$, $xy^2=40$. *Ans.* 10, 20 and 40

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